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ON THE THEORY OF LAMINAR BOUNDARY LAYERS INVOLVING SEPARATION

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SUMMARY

The paper presents a mathematical discussion of the laminar boundary layer, which was developed with a view of facilitating the investigation of those boundary layers in particular for which the phenomenon of separation occurs. The treatment starts with a slight modification of the form of the boundary layer equation first published by von Mises. Two approximate solutions of this equation are found, one of which is exact at the outer edge of the boundary layer while the other is exact at the wall. The final solution is obtained by joining these two solutions at the inflection points of the velocity profiles. The final solution is given in terms of a series of universal functions for a fairly broad class of potential velocity distributions outside of the boundary layer. Detailed calculations of the boundary layer characteristics are worked out for the case in which the potential velocity is a linear function of the distance from the upstream stagnation point. Finally the complete separation point characteristics are determined for the boundary layer associated with a potential velocity distribution made up of two linear functions of the distance from the stagnation point. It appears that extensions of the detailed calculations to more complex potential flows can be fairly easily carried out by using the explicit formulae given in the paper.

1. INTRODUCTION

The theory of the laminar boundary layer has two important applications: First, the computation of skin friction in the range of low Reynolds Numbers; second, the explanation of the separation phenomena. Both problems require the calculation of the "development of the boundary layer" under the assumption of a given pressure distribution along the wall. In some simple cases, as in the case of constant pressure and in that of a pressure decreasing proportionally to a given power of the length measured along the wall, the partial differential equation of the boundary layer theory can be reduced to a total equation and integrated without difficulty. In these cases all "cross sections" of the boundary layer show "similar velocity profiles" and essentially only the scale of the velocity and the length-scale over the cross sections, i.e., the "thickness of the boundary layer" are variable. However, in general, especially in the case in which the pressure is increasing along the wall, the distortion in the shape of the velocity profile is the very point of interest. In these cases the solution of the partial differential equation itself is necessary. Unfortunately, all methods indicated until the present involve such an amount of numerical work as to discourage any engineer anxious to use the theory for solution of practical problems.

K. Pohlhausen (reference 1), at the suggestion of the first author, tried to reduce the boundary layer problem to the solution of a total differential equation in the following way: He chooses a certain family of plausible "velocity profiles" with variable shape in such way that the influence of the rate of change of the pressure along the wall is taken into account by a boundary condition, but one parameter (for instance, the "thickness" of the boundary layer) is left undetermined. Then he applies the momentum law to a strip of the boundary layer enclosed between two adjacent cross sections using the integral relation introduced by the first author. This relation leads to a total differential equation for the undetermined parameter and by integration of this equation the "development of the boundary layer" is obtained.

This method has been criticized especially by von Mises (reference 2), who pointed out that the results depend greatly on the choice of the family of velocity profiles. This is true to some extent, in spite of the fact that in the simple cases mentioned above plausibly chosen velocity profiles lead to close agreement with the exact solution both for the value of the skin friction and the thickness of the boundary layer. Furthermore, the fact that the method can be applied to the case of the turbulent layer, in which we ignore the partial differential equations of the motion, but know the approximate shape of the velocity profile, has to be considered as a great advantage of the method. As a matter of fact, all more or less successful developments in the theory of the turbulent boundary layer are based on the Kármán-Pohlhausen procedure.

However, in the case of the laminar boundary layer, the equations of the motion are known and the difficulties are merely mathematical. Thus a method which contains less arbitrary assumptions than the Pohlhausen procedure and involves less numerical calculations, than the methods used by Hiemenz, Boltze, Thom, Green (reference 3), etc., would represent a desirable improvement of the situation.

The authors became interested in the solution of the boundary-layer equation in connection with their investigations concerning the maximum lift (stalling point) of airfoils. It is generally known that stalling is due to the separation of the boundary layer at the upper surface of an airfoil, but in general it is assumed that the boundary layer is turbulent, except in the immediate neighborhood of the stagnation point, over the range of high Reynolds Numbers corresponding to actual flight or usual wind-tunnel test conditions. Recent investigations in the 10-foot diameter wind tunnel of the Guggenheim Aeronautics Laboratory at the California Institute of Technology have revealed a systematic and very considerable influence of the amount of turbulence in the wind stream on the value of the maximum lift (reference 4). This influence could be correlated very exactly with the influence of the

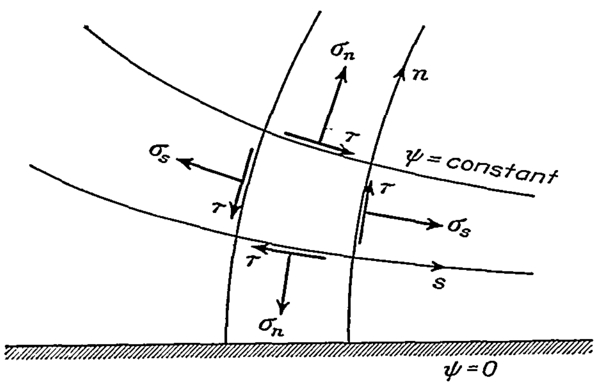

FIGURE 1.—Fluid element in the boundary layer.

turbulence on the so-called "critical" Reynolds Number of a sphere (Reynolds Number characteristic for the sudden drop of the drag coefficient). As L. Prandtl (reference 5) pointed out, the influence of turbulence on the latter phenomenon is due to the fact that the regime in the boundary layer changes earlier from the laminar to the turbulent state, if the turbulence of the wind stream is increased. The authors concluded that a similar process might explain the influence of turbulence on the stalling of airfoils. The conception may be stated as follows: The low values of lift maxima in quiet air streams are due to the separation of the laminar boundary layer, while in a turbulent stream the regime changes before separation occurs and so the lift maximum is raised. For the investigation of such an hypothesis, the calculation of the development of the laminar boundary layer from the stagnation point up to the separation point appeared as a necessity.

This paper presents the method suggested by the authors for the computation of the development of a boundary layer, especially suitable for the case of increasing pressure involving separation. As will be seen, the fundamental equation of motion is used in the form given by von Mises in his paper (reference 2) with a slight modification. This modification furnishes the possibility that the solution of an equation identical with that of ordinary heat conduction, furnishes a first approximation for the velocity field in the region away from the wall. In addition, the shape of the velocity profile near the wall is calculated from the given pressure distribution along the wall. The joining of these two solutions is carried out in a rather rough way, at least from a mathematical point of view. The authors hope that experts in applied mathematics will accomplish the task of improving their somewhat rudimentary procedure.

2. THE FUNDAMENTAL EQUATION IN THE THEORY OF THE BOUNDARY LAYER

All of the following investigations are restricted to the case of 2-dimensional stationary motion. We use a system of curvilinear coordinates. The lines ψ=constant represent stream lines, so that the value of the parameter ψ gives the amount of fluid flowing through between the wall ($\psi=0$) and the particular stream line corresponding to the value of ψ considered. A system of curves perpendicular to the stream lines will be called the n-curves; the length measured along the ψ-curves may be denoted by s, the curvature of the stream lines by k_ψ, that of the n-curves by k_n.

We introduce the following notation:

- u is the magnitude of the velocity, the direction of the velocity being given by the direction of the stream lines.
- p is the pressure.
- σ_s, σ_n, τ are viscous stresses acting on sections normal and parallel to the stream lines (as shown in fig. 1).
- ρ is density of the fluid.
- ν is kinematic viscosity of the fluid$=\mu/\rho$.

Let us consider the equation of momentum applied to the component of the momentum in the direction of the stream lines:

$$\frac{\partial}{\partial s}\left(\frac{u^2}{2}+\frac{p}{\rho}\right)=\frac{1}{\rho}\left(\frac{\partial\sigma_s}{\partial s}+\frac{\partial\tau}{\partial n}-k_\psi\tau+k_n\sigma_n\right).$$

In this equation $\partial/\partial n$ denotes the differential quotient along the n-curves, where the letter n is used for the length measured along the n-curves.

Let us consider the components of the viscous stress σ_s, σ_n, τ. The normal stresses σ_s and σ_n are proportional to the rate of extension of the volume element in the s and n directions, the shearing stress is proportional to the rate of shear. We write

$$\sigma_s=2\mu\frac{\partial u}{\partial s}$$

$$\sigma_n=-2\mu k_n u$$

$$\tau=\mu\left(\frac{\partial u}{\partial n}+k_\psi u\right).$$

Now the theory of the boundary layer is based on two fundamental assumptions:

(*a*) that inside of the boundary layer the differential quotients perpendicular to the direction of flow increase proportionally to $\frac{1}{\sqrt{\nu}}$, as the kinematic viscosity ν tends to zero, while the differential quotients in the direction of the flow remain finite.

(*b*) that the curvature of the stream lines k_ψ remains finite if $\nu \to 0$.

According to these assumptions we can neglect the terms containing σ_s and σ_n in comparison with the terms containing τ, we can neglect $k_\psi \tau$ in comparison with $\frac{\partial \tau}{\partial n}$, and finally we can replace τ by $\mu \frac{\partial u}{\partial n}$

Hence we obtain the relatively simple equation

$$\frac{\partial}{\partial s}\left(\frac{u^2}{2}+\frac{p}{\rho}\right)=\nu \frac{\partial^2 u}{\partial n^2} \tag{1}$$

In this equation p appears as variable both along the stream lines and the n-curves. Following Prandtl's method, we should investigate the variation of p in the direction perpendicular to the flow by considering the equation of momentum applied to the component taken in that direction. According to the assumptions (*a*) and (*b*), we easily obtain the result that the variation of p perpendicular to the streamlines is of the order of $\sqrt{\nu}$. Hence in equation (1) we replace p by its value p_0 corresponding to $n=\infty$, i. e., outside of the boundary layer. In this way p appears as a given function of s.

We may now consider somewhat more exactly the meaning of the two coordinates s and n used in equation (1). s was defined as the length measured along the stream lines. However, it is obvious that with the approximation used in this theory we can use s as the length measured along the wall, and n the length measured along the normal curves (perpendicular to the stream lines) or along a straight line perpendicular to the wall. The approximation used in the theory does not permit of a discrimination between such different definitions of the "cross section of the boundary layer" (reference 2).

We therefore replace the coordinates s and n by two parameters φ and ψ chosen in the following way:

(*a*) Every boundary layer calculation starts from a given "outside flow" in which the influence of viscosity can be neglected, the influence of viscosity being restricted to the boundary layer itself. The velocity of the outside flow at the wall shall be given by $U(s)$. We introduce the parameter φ by the relation $\varphi=\int_0^s U(s)ds$, i.e. φ is the line integral of the "outside velocity" along the wall. If the outside motion is a potential motion, φ is simply the value of the potential along the wall. Obviously we have to write $\frac{\partial}{\partial s}=U\frac{\partial}{\partial \varphi}$.

(*b*) We replace n by the parameter ψ as defined above and write $\frac{\partial}{\partial n}=u\frac{\partial}{\partial \psi}$.

Let us calculate $\frac{\partial^2 u}{\partial n^2}$.

$$\frac{\partial^2 u}{\partial n^2}=\frac{\partial}{\partial n}\left(u\frac{\partial u}{\partial \psi}\right)=u\frac{\partial}{\partial \psi}\left(\frac{\partial \frac{u^2}{2}}{\partial \psi}\right)=u\frac{\partial^2}{\partial \psi^2}\left(\frac{u^2}{2}\right).$$

Hence we write equation (1) in the following form:

$$U\frac{\partial}{\partial \varphi}\left(\frac{u^2}{2}\right)+U\frac{1}{\rho}\frac{dp_0}{d\varphi}=\nu u\frac{\partial^2}{\partial \psi^2}\left(\frac{u^2}{2}\right).$$

Now in the "outside flow" the Bernoulli equation holds, so that $\frac{d}{d\varphi}\left(\frac{p_0}{\rho}+\frac{U^2}{2}\right)=0$ and we obtain:

$$U\frac{\partial}{\partial \varphi}\left(\frac{u^2-U^2}{2}\right)=\nu u\frac{\partial^2}{\partial \psi^2}\left(\frac{u^2}{2}\right),$$

or taking into account the fact that U is a function only of φ and independent of ψ

$$\frac{\partial}{\partial \varphi}\left(\frac{U^2-u^2}{2}\right)=\nu\frac{u}{U}\frac{\partial^2}{\partial \psi^2}\left(\frac{U^2-u^2}{2}\right).$$

We call the quantity $\frac{U^2-u^2}{2}=z$ the "energy defect." It can be interpreted as the loss of energy of a fluid particle of unit mass flowing along a particular stream line as compared with the energy of a particle outside of the boundary layer. Introducing z in the last equation, we obtain

$$\frac{\partial z}{\partial \varphi}=\nu\frac{u}{U}\frac{\partial^2 z}{\partial \psi^2} \tag{2}$$

We notice that this equation is nearly identical with the equation given by von Mises (reference 2). The difference lies essentially in the use of the parameter φ instead of the length s. It will be shown below that this modification facilitates the practical use of the equation.

First it is obvious that the use of φ instead of s does not introduce any complication in the set-up of the problem, because in most cases equally simple expressions are available for the functional relations $\frac{U^2}{2}=f(\varphi)$ and for $U=g(s)$. Second, it is easy to give to equation (2) the following interpretation: In the system φ, ψ, the energy defect $z=\frac{U^2-u^2}{2}$ obeys an equation analogous to the equation of heat conduction. The only coefficient appearing in the equation is the quantity $\nu\frac{u}{U}$, i.e., the kinematic viscosity reduced in the ratio $\frac{u}{U}$, which is the ratio between the local velocity inside the boundary layer and the outside velocity at the same cross section.

Neglecting first the influence of the "reduction factor" $\frac{u}{U}$, i.e., replacing $\nu\frac{u}{U}$ by ν, we obtain a first approximation for the solution of the fundamental equation (2). Because the approximation is most accurate in the outside portion of the boundary layer, where $\frac{u}{U}$ is nearly equal to unity, we will refer to this solution as the "outside solution" and denote it with z_ω. It has the great advantage that it can be obtained by using the well-known methods for integration of the heat-conduction equation.

3. DIMENSIONLESS FORM FOR THE FUNDAMENTAL EQUATION

Before proceeding to the calculation of this first approximation, a dimensionless form will be given for the fundamental equation (2).

We introduce the following notation:

U_o is a suitably chosen characteristic velocity involved in the problem.

L is a suitably chosen characteristic length involved in the problem.

$R=\frac{U_oL}{\nu}$ is the corresponding Reynolds Number.

Then φ^*, ψ^*, u^*, U^*, z^*, shall denote dimensionless quantities defined by the following notation:

$$\left.\begin{aligned}\varphi^*&=\frac{\varphi}{U_oL}\\ \psi^*&=\frac{\psi}{U_oL}\frac{\sqrt{R}}{2}\\ u^*&=\frac{u}{U_o}\\ U^*&=\frac{U}{U_o}\\ z^*&=\frac{z}{U_o^2}\end{aligned}\right\} \tag{3}$$

Substituting these values in equation (2), we obtain

$$\frac{\partial z^*}{\partial \varphi^*}=\frac{1}{4}\frac{u^*}{U^*}\frac{\partial^2 z^*}{\partial \psi^{*2}},$$

where $\frac{u^*}{U^*}$ can be expressed by z^* in the following form:

$$\frac{u^*}{U^*}=\sqrt{1-\frac{2z^*}{U^{*2}}}$$

For convenience, we omit the stars in the succeeding analysis, so that the simple letters denote dimensionless quantities. The same letters are therefore used in the future in a different sense from that in which they were used in the previous sections.

The final equations can now be written:

$$\frac{\partial z}{\partial \varphi}=\frac{1}{4}\frac{u}{U}\frac{\partial^2 z}{\partial \psi^2} \tag{4}$$

and

$$\frac{u}{U}=\sqrt{1-\frac{2z}{U^2}} \tag{5}$$

It is necessary to indicate the boundary conditions for the equation (4). Obviously the value $\psi=0$ corresponds to the wall and $\psi=\infty$ to the transition to the outside flow. Therefore, $u=0$, or $z=\frac{U^2}{2}=z_0$ (say) for $\psi=0$ and $z=0$ for $\psi=\infty$. It is found convenient in the future to use z_0 and $\frac{U^2}{2}$ interchangeably.

4. GENERAL EQUATIONS FOR THE FIRST APPROXIMATION

(OUTSIDE SOLUTION)

Let us consider the equation

$$\frac{\partial z_\omega}{\partial \varphi}=\frac{1}{4}\frac{\partial^2 z_\omega}{\partial \psi^2} \tag{6}$$

which we obtain from (4) replacing $\frac{u}{U}$ by 1, or neglecting $\frac{2z}{U^2}$ in (5) in comparison with 1, and writing $z=z_\omega$.

This approximation cannot in general hold near the wall, where $\frac{2z}{U^2}$ approaches the value 1. However, it is not without interest to discuss this simple approximate solution, which we shall later use for the outside part of the boundary layer.

Considering z as "temperature", and replacing φ by the time t, ψ by a reduced length-coordinate x, the following analogous heat-conduction problem can be formulated: The temperature at the end point $x=0$ of a bar of infinite length is a given function of the time t beginning at $t=0$. At infinite distance ($x=\infty$) the temperature is equal to zero. The temperature distribution as a function of time and distance from $x=0$ is to be computed.

For either heat conduction or boundary layer statement of the problem, we have to find the solution of (6), which satisfies the boundary conditions $z_\omega=0$ for $\psi=\infty$ and $z_\omega=z_0=\frac{U^2(\varphi)}{2}$ for $\psi=0$. This solution is given by the definite integral

$$z_\omega=\frac{\psi}{2\sqrt{\pi}}\int_0^\varphi \frac{e^{-\frac{\psi^2}{\varphi-\xi}}}{(\varphi-\xi)^{3/2}}U^2(\xi)d\xi \tag{7}$$

which is given in any text on the theory of heat conduction. Hence with this approximation the velocity distribution in the boundary layer can be computed by quadratures, as U^2 is a given function of the outside potential φ.

It is interesting that a relatively simple explicit condition can be found for the location of the separation point. Separation occurs if the slope of the velocity at the wall changes sign. Thus the condition for separation can be written in the form $\frac{\partial z}{\partial \psi}=0$ for $\psi=0$. We obtain by differentiation

$$\frac{\partial z_\omega}{\partial \psi}=\frac{1}{2\sqrt{\pi}}\int_0^\varphi \frac{e^{-\frac{\psi^2}{\varphi-\xi}}}{(\varphi-\xi)^{3/2}}U^2(\xi)\mathrm{d}\xi-\frac{\psi^2}{\sqrt{\pi}}\int_0^\varphi \frac{e^{-\frac{\psi^2}{\varphi-\xi}}}{(\varphi-\xi)^{5/2}}U^2(\xi)\mathrm{d}\xi.$$

For $\psi=0$ the second term vanishes, and the first term becomes indeterminate, but its value can be determined by a partial integration. In this way we obtain

$$\left(\frac{\partial z_\omega}{\partial \psi}\right)_{\psi=0}=-\frac{2}{\sqrt{\pi}}\int_0^\varphi \frac{U\frac{\mathrm{d}U}{\mathrm{d}\xi}}{\sqrt{\varphi-\xi}}\mathrm{d}\xi.$$

Thus the separation criterion is given as $\varphi=\varphi_s$ where

$$F(\varphi_s)=\int_0^{\varphi_s}\frac{U\frac{\mathrm{d}U}{\mathrm{d}\xi}}{\sqrt{\varphi_s-\xi}}\mathrm{d}\xi=0. \tag{8}$$

Assuming $\varphi=0$ as stagnation point ($U=0$) and assuming positive values for U, we easily see that separation can only occur after $\frac{\mathrm{d}U}{\mathrm{d}\xi}$ changes sign, i.e., after the velocity U has reached a maximum value and is decreasing.

We will find later that this separation condition is not of great practical value, because as we approach the separation point, the approximate solution becomes very inexact near the wall.

5. THE VELOCITY PROFILE AND DIFFERENTIAL EQUATION IN TERMS OF $u=u(x, y)$

The differential equation (6) determines the energy defect $z=\frac{U^2-u^2}{2}$ as function of the parameters φ and ψ. For many applications, expressions for the velocity profile are needed, i.e., one has to determine the velocity u as function of the coordinates x and y, where x is the distance of an arbitrary cross section from the starting point of the boundary layer, and y is the distance from the wall. Instead of the real lengths x and y, dimensionless quantities x^* and y^* may be defined in terms of the characteristic length L, putting

$$x^*=\frac{x}{L}\quad y^*=\frac{y}{L}$$

For convenience we use the letters x and y instead of the letters x^* and y^*, as we did above for the other quantities involved in this investigation. Then it is obvious from the definitions given in equation (3) that u, x, y can be expressed by the formulas:

$$\left.\begin{aligned}
u&=\sqrt{U^2-2z}=U\sqrt{1-\frac{z}{z_0}}\\
x&=\int_0^\varphi\frac{\mathrm{d}\varphi}{U}=\int_0^\varphi\frac{\mathrm{d}\varphi}{\sqrt{2z_0}}\\
y&=\frac{2}{\sqrt{R}}\int_0^\psi\frac{\mathrm{d}\psi}{u}=\frac{2}{U\sqrt{R}}\int_0^\psi\frac{\mathrm{d}\psi}{\sqrt{1-\frac{z}{z_0}}}
\end{aligned}\right\} \tag{9}$$

These formulas enable us to calculate the velocity profiles corresponding to any given solution $z(\varphi, \psi)$.

It is interesting to transform the approximate equation (6) into an equation for u, containing x and y as independent variables. Obviously the symbol $\partial/\partial\varphi$ means in this case the rate of change of any quantity with increasing x while keeping constant y, while $\partial/\partial\varphi$ means differentiation by keeping ψ invariable. Therefore

$$\frac{\partial}{\partial\varphi}=\frac{1}{U}\left(\frac{\partial}{\partial x}+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial}{\partial y}\right),$$

where $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the inclination of the line $\psi=$constant in the x, y coordinate system. The inclination $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be expressed by

$$-\frac{\frac{\partial\psi}{\partial x}}{\frac{\partial\psi}{\partial y}}=\frac{v}{u},$$

where v is the (dimensionless) velocity in the y direction. We obtain in this way

$$\frac{\partial z_\omega}{\partial\varphi}=\frac{1}{U}\left(\frac{\partial z_\omega}{\partial x}+\frac{v}{u}\frac{\partial z_\omega}{\partial y}\right)=\frac{\mathrm{d}U}{\mathrm{d}x}-\frac{1}{U}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right).$$

On the other hand,

$$\frac{\partial^2 z}{\partial\psi^2}=-\frac{\partial}{\partial\psi}\left(u\frac{\partial u}{\partial\psi}\right)=\frac{-2}{\sqrt{R}}\frac{\partial}{\partial\psi}\left(\frac{\partial u}{\partial y}\right)=\frac{-4}{Ru}\frac{\partial^2 u}{\partial y^2}$$
(from (9)).

The equation (6) is therefore transformed into

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{R}\frac{U}{u}\frac{\partial^2 u}{\partial y^2}+U\frac{\mathrm{d}U}{\mathrm{d}x}.$$

Calling the dimensional quantities u_1, v_1, U_1, x_1, and y_1 for the moment, we obtain

$$u_1\frac{\partial u_1}{\partial x_1}+v_1\frac{\partial u_1}{\partial y_1}=\nu\frac{U_1}{u_1}\frac{\partial^2 u_1}{\partial {y_1}^2}+U_1\frac{\mathrm{d}U_1}{\mathrm{d}x_1} \tag{10}$$

Comparing equation (10) with the "exact" boundary layer equation deduced first by L. Prandtl:

$$u_1\frac{\partial u_1}{\partial x_1}+v_1\frac{\partial u_1}{\partial y_1}=\nu\frac{\partial^2 u_1}{\partial {y_1}^2}+U_1\frac{\mathrm{d}U_1}{\mathrm{d}x_1}, \tag{11}$$

we notice that equation (10) exaggerates the influence of the term corresponding to the viscous friction, multiplying the term $\nu \frac{\partial^2 u_1}{\partial y_1^2}$ with the ratio $\frac{U}{u}$. We especially notice that for $y=0$, i.e., at the wall, $u=0$ and so $\frac{U}{u}\frac{\partial^2 u}{\partial y^2}$ can only have finite value if $\frac{\partial^2 u}{\partial y^2}=0$. Hence the solution of equation (6), called in this paper the "outer solution", is composed of velocity profiles with vanishing second differential quotients $\frac{\partial^2 u}{\partial y^2}$ at the wall. This is the reason that it cannot be expected that S shape velocity profiles, such as occur in a flow against increasing pressure before separation, will be satisfactorily represented by the solution of the approximate equation (6).

6. POWER SERIES EXPANSION OF $U^2(\varphi)$

The analytical evaluation of the integral in (7) for an arbitrary function $U^2(\varphi)$ is in general difficult, if not impossible. Numerical or graphical methods are likewise of little use. We therefore adopt the procedure of approximating to the function $U^2(\varphi)$ by a polynomial of the form

$$U^2(\varphi)=\sum_{i=0}^{n} b_i\varphi^i \tag{12}$$

With such a function the indicated integrations can be analytically carried out. In proceeding with the calculations to this end, it is convenient first to transform the expression for z_ω in (7) by introducing a new variable of integration defined by

$$\beta=\frac{\psi}{\sqrt{\varphi-\xi}}$$

A little calculation gives at once

$$z_\omega=\frac{1}{\sqrt{\pi}}\int_{\frac{\psi}{\sqrt{\varphi}}}^{\infty} e^{-\beta^2}U^2\left(\varphi-\frac{\psi^2}{\beta^2}\right)d\beta, \tag{13}$$

or introducing the particular function U^2 given in (12):

$$z_\omega=\frac{1}{\sqrt{\pi}}\sum_{i=0}^{n} b_i\int_{\frac{\psi}{\sqrt{\varphi}}}^{\infty} e^{-\beta^2}\left(\varphi-\frac{\psi^2}{\beta^2}\right)^i d\beta.$$

Expanding the term in parenthesis by the binomial theorem and integrating term by term, we obtain:

$$z_\omega=\frac{b_0}{2}J\left(\frac{\psi}{\sqrt{\varphi}}\right)+\frac{1}{2}\sum_{i=1}^{n} b_i\varphi^i\sum_{r=0}^{i}(-)^r\frac{i!}{(i-r)!r!}\left(\frac{\psi^2}{\varphi}\right)^r J_r\left(\frac{\psi}{\sqrt{\varphi}}\right),$$

where

$$J_r(x)=\frac{2}{\sqrt{\pi}}\int_x^{\infty}\frac{e^{-\beta^2}}{\beta^{2r}}d\beta, \text{ and } 0!=1.$$

By repeated partial integrations J_r where r is any integer greater than zero, may be expressed in terms of J_o. Carrying out the partial integrations and collecting terms, we obtain, finally:

$$\left.\begin{aligned}
&z_\omega(\varphi,\psi)=b_0g_0\left(\frac{\psi}{\sqrt{\varphi}}\right)+\sum_{i=1}^{n} b_i\varphi^i\sum_{r=0}^{i}\frac{i!}{(i-r)!\,r!}g_r\left(\frac{\psi}{\sqrt{\varphi}}\right)\\
&\text{where}\\
&g_0(x)=\frac{J_0(x)}{2}\\
&g_1(x)=-\frac{1}{\sqrt{\pi}}xe^{-x^2}+x^2J_0(x)\\
&g_r(x)_{r\geqq 2}=\frac{(r-1)!}{(2r-1)!}\left[\frac{(-)^r}{\sqrt{\pi}}xe^{-x^2}\sum_{k=0}^{r-2}(-)^k2^{2k+1}\frac{(2r-2k-3)!}{(r-k-2)!}x^{2k}+2^{2(r-1)}\left\{\frac{-1}{\sqrt{\pi}}xe^{-x^2}x^{2(r-1)}+J_0(x)x^{2r}\right\}\right]\\
&J_0(x)=1-\frac{2}{\sqrt{\pi}}\int_0^x e^{-\beta^2}d\beta, \text{ and } 0!=1.
\end{aligned}\right\} \tag{14}$$

The universal functions $g_r(x)$ can very readily be calculated using the standard tables of the probability integral appearing in $J_o(x)$. Once determined they are independent of the velocity function $U(\varphi)$, provided only that the latter has the polynomial form (12). Equation (14) therefore, in connection with (9), gives the complete analytical solution for our first approximation to the boundary layer problem for any case in which the potential velocity is such that U^2 may be expressed as a polynomial in φ.

Substituting (12) in the separation criterion (8), integrating term by term, and again applying successive partial integrations, it is very easy to show that the separation criterion takes the form, where φ_s is the value of φ at the separation point:

$$F(\varphi_s)=\sum_{i=1}^{n}2^{2i-1}\frac{i!(i-1)!}{(2i-1)!}b_i\varphi_s^{i-1}=0 \tag{15}$$

Hence the determination of the separation point is reduced to the solution of an algebraic equation with given numerical coefficients.

7. DOUBLE POWER SERIES EXPANSION OF $U^2(\varphi)$

In principle we can approximate with any desired exactness to any practically important velocity function $U^2(\varphi)$ by a polynomial of the form (12). For many important cases, however, such as, for example, the potential velocity around an airfoil, the number of terms required in the polynomial for a satisfactory approximation becomes very large. Since the expression for z_ω becomes rapidly more and more complicated as the number of terms in the polynomial increases, it follows that the solution of our problem becomes very laborious and complex in such cases. For this reason a solution has been investigated for the case in which $U^2(\varphi)$ is approximated by two distinct polynomials of the form (12) instead of by one. This solution, which is discussed in the present section, is somewhat more complicated formally than is (14), but is vastly simpler for carrying out actual calculations.

We approximate to $U^2(\varphi)$ by two polynomials, one of which we use for $\varphi\leqq\varphi_1$ and one for $\varphi\geqq\varphi_1$, where φ_1 is any fixed value of ϕ, i.e.:

$$\left.\begin{aligned} U^2(\varphi)&=f_1(\varphi)=\sum_{i=0}^{n} b_i\varphi^i \text{ for } \varphi\leqq\varphi_1 \\ U^2(\varphi)&=f_2(\varphi)=\sum_{i=0}^{m} \beta_i\varphi^i \text{ for } \varphi\geqq\varphi_1 \end{aligned}\right\} \quad (16)$$

The procedure is indicated schematically in figure 2, in which the solid line represents the $U^2(\varphi)$ actually used.

We substitute (16) in the original integral for z_ω, change the variable of integration, expand, integrate term by term, and perform successive partial integrations just as in the analysis leading to (14). The final result may be written in the form:

$$\left.\begin{aligned} &z_\omega(\varphi,\ \psi)=z_b-z_b^*+z_\beta^*(\varphi\geqq\varphi_1) \\ &\text{where} \\ &z_b=z_\omega \text{ of formula (14)} \\ &z_b^*=b_0g_0\left(\frac{\psi}{\sqrt{\varphi-\varphi_1}}\right) \\ &+\sum_{i=1}^{n}b_i\sum_{r=0}^{i}\frac{i!}{(i-r)!r!}g_r\left(\frac{\psi}{\sqrt{\varphi-\varphi_1}}\right)\varphi^{i-r}(\varphi-\varphi_1)^r \end{aligned}\right\} \quad (17)$$

z_β^* is identical with z_b^* except $b_i\longrightarrow\beta_i$ and $n\longrightarrow m$. $0!=1$ and $g_r(x)$ is as given in equation (14). If $\varphi\leqq\varphi_1$ we use the simpler formula (14).

Proceeding as before, the separation criterion now takes the form

$$\left.\begin{aligned} F(\varphi_s)=\sum_{i=1}^{n}\frac{2^{2i-1}i!(i-1)!}{(2i-1)!}b_i\varphi_s^{i-1}+ \\ +\sqrt{1-\frac{\varphi_1}{\varphi_s}}\sum_{i=1}^{n\text{ or }m}\frac{i!(i-1)!}{(2i-1)!}(\beta_i-b_i)\sum_{r=1}^{i}\frac{2^{2r-1}(2i-2r)!}{[(i-r)!]^2} \\ \varphi_s^{r-1}\varphi_1^{i-r}=0 \end{aligned}\right\} \quad (18)$$

$0!=1$, the second sum is to be taken to the larger of n or m, and $b_i=0$ for $i>n$, $\beta_i=0$ for $i>m$.

For many practically important cases it is sufficiently accurate to approximate to $U^2(\varphi)$ by two power series involving no terms of degree higher than 3 in φ, i.e., $b_4=b_5=\text{---------}=\beta_4=\beta_5=\text{---------}=0$. In such cases the formula (17) can be written in the following form, which is very convenient for purposes of calculation:

$$\left.\begin{aligned} z_\omega(\varphi,\ \psi)=\{&b_0(h_0-h_0^*)+\beta_0h_0^*+\gamma_1\varphi_1g_1^* \\ &-\gamma_2\varphi_1^2g_2^*+\gamma_3\varphi_1^3g_3^*\} \\ +\varphi\{&b_1(h_1-h_1^*)+\beta_1h_1^*+2\gamma_2\varphi_1(g_1^* \\ &+g_2^*)-3\gamma_3\varphi_1^2(g_2^*+g_3^*)\} \\ +\varphi^2\{&b_2(h_2-h_2^*)+\beta_2h_2^*+3\gamma_3\varphi_1(g_1^* \\ &+2g_2^*+g_3^*)\} \\ +\varphi^3\{&b_3(h_3-h_3^*)+\beta_3h_3^*\} \qquad (\textit{for } \varphi\geqq\varphi_1) \\ \text{where}& \\ \gamma_i=b_i-\beta_i,\ g_i&=g_i\left(\frac{\psi}{\sqrt{\varphi}}\right),\ g_i^*=g_i\left(\frac{\psi}{\sqrt{\varphi-\varphi_1}}\right), \\ h_0=g_0,\ h_1=g_0+g_1,&\ h_2=g_0+2g_1+g_2,\ h_3=g_0+3g_1 \\ +3g_2+g_3,&\ h_0^*=g_0^*,\ h_1^*=g_0^*+g_1^*,\ \textit{etc.} \\ \text{If } \varphi\leqq\varphi_1 &\textit{ put } \beta_1=g_1^*=h_1^*=0 \end{aligned}\right\} \quad (19)$$

Before proceeding with the general case, it is interesting to compare this approximation with the exact

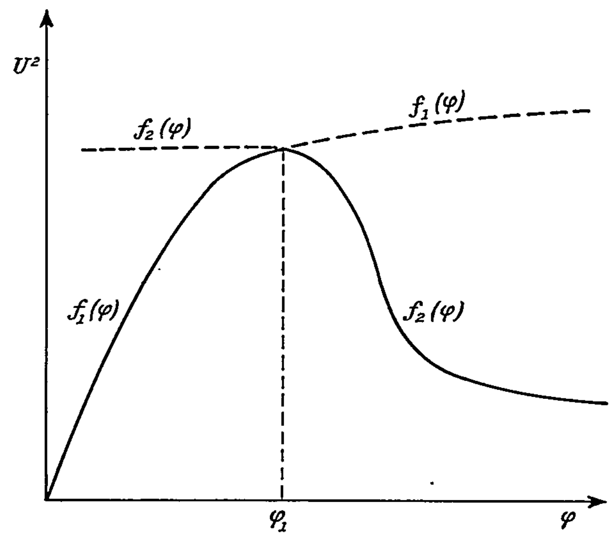

FIGURE 2.—Double power series representation of $U^2(\varphi)$.

solution given by Blasius for uniform external flow along a flat plate.

8. APPLICATION TO BLASIUS' CASE U=constant

For this simple case, we have in accordance with (16)

$$U^2=b_0,\ b_1=b_2=\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ =\beta_i=0,\ \varphi_1=\infty$$

Hence from (19) and (14)

$$z_\omega(\varphi,\psi)=b_0h_0=b_0g_0=\frac{b_0}{2}\left[1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{\psi}{\sqrt{\varphi}}}e^{-\beta^2}\,d\beta\right]$$

If we take $U=U_0$=the characteristic velocity used in defining the Reynolds Number, then U has the numerical value 1, i.e.:

$$z_\omega(\varphi,\psi)=\frac{1}{2}\left[1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{\psi}{\sqrt{\varphi}}}e^{-\beta_2}\,d\beta\right]$$

For convenience, we denote the probability integral by P, i.e.:

$$P(x)=\frac{2}{\sqrt{\pi}}\int_0^{x}e^{-\beta^2}\,d\beta$$

Then from the basic equations (9), we have for the dimensionless velocity and distance from the wall

$$\left.\begin{aligned} u &= \sqrt{P\left(\frac{\psi}{\sqrt{\varphi}}\right)} \\ y &= \frac{2\sqrt{\varphi}}{\sqrt{R}}\int_0^{\frac{\psi}{\sqrt{\varphi}}} \frac{d\beta}{\sqrt{P(\beta)}} \end{aligned}\right\} \tag{20}$$

We arbitrarily define the boundary layer thickness δ as that value of y at which the velocity has 99.5 percent of the value which it has in the outside potential flow, i.e.,

$$y=\delta \sim u=0.995 \tag{21}$$

From the tables of the probability integral P, we find that this occurs for $\frac{\psi}{\sqrt{\varphi}}=1.8$.

Hence
$$\delta=\frac{2\sqrt{\varphi}}{\sqrt{R}}\int_0^{1.8}\frac{d\beta}{\sqrt{P(\beta)}}$$

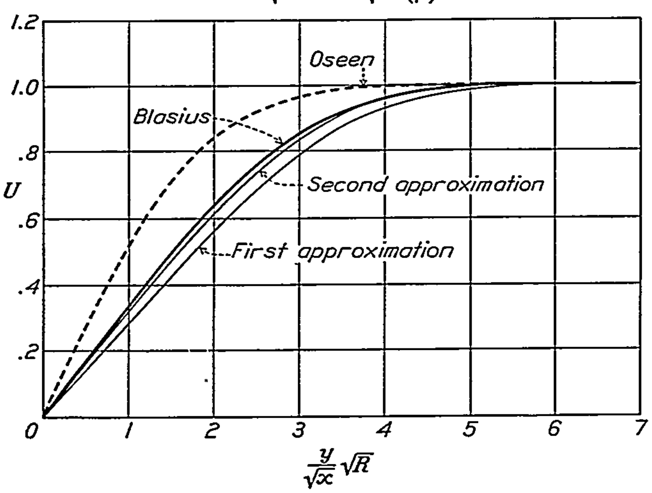

FIGURE 3.—Dimensionless velocity profiles for flow along a flat plate.

The evaluation of the integral in (20) has been carried out analytically for $0 \leq \frac{\psi}{\sqrt{\phi}} \leq 0.1$ using the standard power series expansion:

$$P(x)=\frac{2}{\sqrt{\pi}}\left(x-\frac{x^3}{3}+\frac{x^5}{10}+\cdots\cdots\right).$$

For $\frac{\psi}{\sqrt{\varphi}} \geq 0.1$ the integration has been accomplished graphically with a Coradi integraph. Before presenting the results it will be convenient to introduce the distance along the plate in a direction parallel to U and measured from the leading edge, which is also taken as the origin of φ. In order to fix ideas, we consider a plate whose total length in the direction of U is taken as the characteristic length L which occurs in the Reynolds Number. Then in order to make our distance along the plate a dimensionless variable, we divide it by L. This dimensionless distance we denote by x in accordance with customary practice, so that $0 \leq x \leq 1$. For the present case $\varphi=Ux=x$ since $U=1$. Hence φ may be replaced throughout by x.

The results for $u=u(y)$ are given in figure 3 by the curve labeled "first approximation". The Blasius profile is also plotted for comparison. The expression for δ from this first approximation is $\delta=\frac{5.53}{\sqrt{R}}\sqrt{x}$, while the usual Blasius value is $\delta_{\text{Blasius}}=\frac{5.5}{\sqrt{R}}\sqrt{x}$.

Here the dimensionless length δ is, of course, also defined in terms of the standard length L. The shearing stress at the wall τ_0, as obtained from $\left(\frac{\partial u}{\partial y}\right)_{\text{wall}}$ has the following expressions:

$$\left(\frac{\tau_0}{\rho U^2}\right)_{\text{1st approx.}}=\frac{0.282}{\sqrt{Rx}},\ \left(\frac{\tau_0}{\rho U^2}\right)_{\text{Blasius}}=\frac{0.332}{\sqrt{Rx}}$$

It has been pointed out above that our first approximation corresponds to an exaggeration of the viscous forces near the wall. The method of linearizing the differential equation introduced by Oseen retains the exact viscous terms but approximates to the inertia terms. It is of some interest, therefore, to compare the results given by the two methods. This is especially easy for the present case of flow without pressure drop, since the Oseen method leads to an equation of exactly the same nature as does our first approximation. The Oseen solution has, therefore, been worked out and will be briefly discussed here.[1]

We start with the dimensionless form of the Prandtl equation which, for U=constant, becomes (cf. equation (11)):

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}.$$

We then introduce the velocity defect u_2 defined by:

$$u_2=U-u$$

Substituting this in the Prandtl equation, the Oseen method of linearizing is based on the neglect of all second-degree terms in u_2 or v. Carrying out this procedure, we obtain

$$\frac{\partial u_2}{\partial x}=\frac{1}{UR}\frac{\partial^2 u_2}{\partial y^2},$$

i.e., a heat-conduction equation of exactly the type as was deduced for our first approximation (6). The boundary conditions are: $u_2=U=1$ at $y=0$ and $u_2=0$ at $y=\infty$. The solution corresponding to these boundary conditions is obtainable at once in terms of the probability integral as

$$u_2=1-UP\left(\frac{\sqrt{RU}}{2}\frac{y}{\sqrt{x}}\right)$$

or in terms of the original variable u (since $U=1$)

[1] After the present paper was completed, the authors discovered that the Oseen method results presented in the remainder of this section had been investigated from a point of view rather different from theirs and presented in a paper by N. A. V. Piercy and H. F. Winny "The Skin Friction of Flat Plates to Oseen's Approximation", Proc. Roy. Soc. Lond., Series A, Volume 140 (1933), p. 543.

$$\frac{u}{U}=P\left(\frac{\sqrt{R}}{2}\frac{y}{\sqrt{x}}\right).$$

This velocity profile has also been plotted in figure 3 and labeled "Oseen." The boundary-layer thickness defined as above is given by

$$\delta_{Oseen}=\frac{4.0}{\sqrt{R}}\sqrt{x},$$

while the velocity gradient at the wall can readily be shown to be just twice that given by our first approximation, so that

$$\left(\frac{\tau_0}{\rho U^2}\right)_{Oseen}=\frac{0.564}{\sqrt{Rx}}.$$

Comparing all of these results, we see that our first approximation gives excellent agreement with the Blasius exact solution as far as δ is concerned, and a fairly satisfactory agreement with respect to velocity profile and wall shearing stress. The Oseen method of linearizing the differential equation leads, in the present case, to much greater errors in all of these characteristics.

9. FURTHER APPROXIMATION TO THE OUTER SOLUTION

In order to improve the accuracy of our solution, it is natural to look for some method of successive approximation or iteration starting with the first approximate solution discussed above. Von Mises (reference 2) has suggested a method based essentially on successive iterations in the φ direction. If we use subscripts 1 to denote our first approximation and subscripts 2 for a second approximation, then in view of the form of the exact equations (4), (5), von Mises suggests essentially that we write

$$z_2(\varphi,\psi)=\frac{1}{4}\int_0^{\varphi}\sqrt{1-\frac{2z_1}{U^2}}\frac{\partial^2 z_1}{\partial\psi^2}d\varphi.$$

This procedure has been investigated for the Blasius case and found to lead to inconsistencies with the given boundary conditions of the problem. The difficulties seemed to be basic ones, so that this method of approximation was abandoned and the following alternative procedure was considered:

The exact differential equation may be written in the form

$$\frac{\partial z}{\partial\varphi}-\frac{1}{4}\frac{\partial^2 z}{\partial\psi^2}=\frac{1}{4}\frac{\partial^2 z}{\partial\psi^2}\left(\sqrt{1-\frac{2z}{U^2}}-1\right)$$

Then the first approximation is obtained by neglecting the right side of the equation, i.e.:

$$\frac{\partial z_\omega}{\partial\psi}-\frac{1}{4}\frac{\partial^2 z_\omega}{\partial\psi^2}=0.$$

The following second approximation immediately suggests itself:

$$\frac{\partial z_2}{\partial\varphi}-\frac{1}{4}\frac{\partial^2 z_2}{\partial\psi^2}=\frac{1}{4}\frac{\partial^2 z_\omega}{\partial\psi^2}\left(\sqrt{1-\frac{2z_\omega}{U^2}}-1\right)=f_1(\varphi,\psi),\text{ say.}$$

Then z_2 is the solution of a generalized, nonhomogeneous, heat-conduction equation, in which the right side f_1, is a given function of φ, ψ. From any standard work on mathematical analysis[1]), the formal solution of this equation satisfying the given boundary conditions can readily be found to be

$$z_2(\varphi,\psi)=z_\omega(\varphi,\psi)-\frac{1}{\sqrt{\pi}}\int_0^\infty d\eta\int_0^\varphi d\xi\left[e^{-\frac{(\psi-\eta)^2}{\varphi-\xi}}-e^{-\frac{(\psi+\eta)^2}{\varphi-\xi}}\right]\frac{f_1(\xi,\eta)}{\sqrt{\varphi-\xi}}$$

Unfortunately, the authors have found the evaluation of this integral form of solution too intractable for the method to be useful for the case of any general function $U(\varphi)$. Only in the Blasius case, where a separation of variables is possible so that the partial differential equation reduces to an ordinary differential equation with independent variable $\frac{\psi}{\sqrt{\varphi}}$, has a solution of this second approximation been calculated. Since the method is not pursued further in the present paper, the analysis leading to this solution is not presented. The velocity profile so deduced is, however, included in figure 3, with the label "second approximation". The approximation to the exact solution is seen to be quite good.

In spite of this initial success the entire method of successive approximations was abandoned for the reasons mentioned above. It appears, however, that it might furnish an interesting and perhaps profitable field of study for the mathematician.

10. THE INNER SOLUTION

It has already been pointed out that the approximate form of the boundary-layer equation which leads to the outer solution discussed above is exact at the outer edge of the boundary layer, but is in general far from correct at the wall. A natural procedure is therefore to look for another approximation to the exact equation which shall be correct at the wall but possibly inaccurate far from the wall. The solution of this equation might naturally be called the "inner solution", and the final method of approximating to the exact and complete problem would then be to join the outer and inner solutions together in some way.

A first approximation to this inner solution is very easily obtained, but it is convenient to first change the notation slightly in setting up the equation. We introduce a new independent variable defined by

$$\zeta=z_0-z=\frac{u^2}{2}\text{ where }z_0=\frac{U^2}{2}. \tag{22}$$

Then the basic differential equation (4) may be written in the form

$$\frac{\partial^2\zeta}{\partial\psi^2}=\frac{-4z_0'\sqrt{z_0}}{\sqrt{\zeta}}\left(1-\frac{1}{z_0'}\frac{\partial\zeta}{\partial\varphi}\right) \tag{23}$$

[1] cf. for example, Goursat's "Cours d'Analyse III," ch. XXIX.

where

$$z_0' = \frac{d}{d\varphi}\left(\frac{U^2}{2}\right).$$

Using a subscript $(\ )_i$ to denote the inner solution, we approximate to (23) by writing

$$\left.\begin{aligned}\frac{\partial^2 \zeta_i}{\partial \psi^2} &= \frac{A(\varphi)}{\sqrt{\zeta_i}}\left(1-\frac{1}{z_0'}\left(\frac{\partial \zeta}{\partial \varphi}\right)^*\right)\\ A(\varphi) &= -4z_0'\sqrt{z_0}\end{aligned}\right\} \quad (24)$$

where $\left(\frac{\partial \zeta}{\partial \varphi}\right)^*$ is a function of φ, ψ, ζ_i chosen in such a way that (24) approaches (23) as the distance from the wall goes to zero, i.e., as $\psi \rightarrow 0$.

Since at the wall we have $\zeta = \frac{\partial \zeta}{\partial \varphi} = 0$, it appears that to get a first approximation for our inner solution we may take $\left(\frac{\partial \zeta}{\partial \varphi}\right)^* = 0$. This solution, denoted by ζ_{i_1} will then be exact at the wall. Discussing this simple approximation, we have

$$\frac{\partial^2 \zeta_{i_1}}{\partial \psi^2} = \frac{A(\varphi)}{\sqrt{\zeta_{i_1}}} \quad (25)$$

We consider this as an ordinary differential equation and integrate for φ=constant. It is convenient to introduce temporarily the auxiliary dependent variable $W = \left(\frac{\partial \zeta_{i_1}}{\partial \psi}\right)^2$, and to consider ζ_{i_1} as independent variable. Then we obtain at once

$$\frac{1}{2}\frac{dW}{d\zeta_{i_1}} = \frac{A}{\sqrt{\zeta_{i_1}}},$$

or integrating,

$$W(\zeta_{i_1}) - W(0) = 4A\sqrt{\zeta_{i_1}}.$$

Now writing

$$B_1^2(\varphi) = W(0) = \left(\frac{\partial \zeta_{i_1}}{\partial \psi}\right)^2_{\psi=0} \sim \left(\frac{\partial u}{\partial y}\right)^2_{y=0}$$

and returning to the original variables ζ_{i_1} and ψ,

$$\frac{\partial \zeta_{i_1}}{\partial \psi} = [B_1^2 + 4A\sqrt{\zeta_{i_1}}]^{1/2}.$$

A little calculation gives finally, assuming $\zeta_{i_1}(\psi=0)=0$.

$$\psi = \frac{1}{6A^2}\left[B_1^3 - (B_1^2 - 2A\sqrt{\zeta_{i_1}})\sqrt{B_1^2 + 4A\sqrt{\zeta_{i_1}}}\right]. \quad (26)$$

If $B_1 \neq 0$, we may invert this expression obtaining the following expansion about $\psi=0$:

$$\zeta_{i_1} = B_1\psi + \frac{4}{3}\frac{A}{\sqrt{B_1}}\psi^{3/2} + \ldots \quad (B_1 \neq 0). \quad (27)$$

At the separation point $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$ and hence $B_1^2 = 0$. We have exactly, therefore:

$$\left.\begin{aligned}\psi &= \frac{2}{3\sqrt{A}}\zeta_{i_1}^{3/4} \text{ or}\\ \zeta_{i_1} &= \left(\frac{3}{2}\right)^{4/3} A^{2/3}\psi^{4/3}\end{aligned}\right\} \textit{At separation point: } B_1 = 0 \quad (28)$$

The different analytical form for ζ_{i_1}, at and away from the separation point for $\psi \rightarrow 0$ is very interesting. In view of the exactness of the equation for ζ_{i_1}, at $\psi=0$, any further approximations for ζ_i must exhibit the same behavior near $\psi=0$ as is shown in (27) and (28).

11. THE IMPROVED INNER SOLUTION AND ITS JOINING TO THE OUTER SOLUTION

In developing further approximations to the inner solution, we must be guided by the procedure which will be used in joining the two solutions together. A discussion of this procedure requires first a brief reconsidering of the approximate outer solution. The exact and approximate outer equations are, respectively:

$$\frac{\partial z}{\partial \varphi} = \frac{1}{4}\frac{\partial^2 z}{\partial \psi^2}\left(1-\frac{z}{z_0}\right)^{1/2} \text{ and } \frac{\partial z_\omega}{\partial \varphi} = \frac{1}{4}\frac{\partial^2 z_\omega}{\partial \psi^2}.$$

The two equations are identical at the outer edge of the boundary layer where $z=0$ and would at first glance appear to become more and more different as the wall is approached, since z in general increases continuously from 0 to z_0. Fortunately, however, the two equations are again identical when $\frac{\partial^2 z}{\partial \psi^2}=0$, i.e., at an inflection point of any $z(\psi)$ profile. For the Blasius case of no pressure drop this inflection point occurs just at the wall. For external flows with pressure drop in the direction of U it is almost certain that there is no inflection point at all. For flows with pressure increase in going downstream the inflection point moves out from the wall so that the $z(\psi)$ profiles develop an S-shape. In view of the nature of the two equations, it therefore appears that the outer solution furnishes, in most cases, a satisfactory approximation to the exact solution for the region between the outer edge of the boundary layer and the inflection points of the $z(\psi)$ profiles. Proceeding from these inflection points to the wall the accuracy of the outer solution apparently becomes rapidly more and more unsatisfactory. The procedure adopted in view of this situation is the following: For flows with pressure decrease or constant pressure the outer solution is considered as a satisfactory approximation to the exact solution. For flows with pressure increase the outer solution is used from the outside of the boundary layer to the curve connecting the inflection points of the outer solution $z_\omega(\psi)$ profiles. For the region between this curve and the wall the partial differential equation is replaced by a family of ordinary differential equations in ψ. The inner solution is composed of the functions $\zeta_i(\psi)$ obtained by the solution of these equations and joined to the appropriate $\zeta_\omega(\psi)$ profile of the outer solution for each value of φ. In the following discussion of the inner solution and its joining to the outer solution, we are therefore considering only flows with pressure increase in the direction of the external flow.

In joining the two solutions it seems necessary to require that $\frac{\partial^2\zeta}{\partial\psi^2}$ be continuous, in view of the major importance of this second derivative in the equations. We must, therefore, join the two solutions in such a way that the inflection points of the outer solution coincide with inflection points of the inner solution. Referring to equation (25), we see that for a given value of φ, $\frac{\partial^2\zeta_{i_1}}{\partial\psi^2}$ has always the same sign for all values of ψ. In other words, the first approximate inner solution has no inflection points and is, therefore, quite useless for our present purpose. Some 15 different procedures for finding a satisfactory inner solution and joining it to the outer one were investigated in all, the more important of which will be briefly discussed below.

It is convenient to call the point at which the inner and outer solutions are joined the "joining point" and to use a subscript j to denote quantities taken at this point. It is also convenient to introduce an alternative independent variable $\xi=\frac{\psi}{\sqrt{\varphi}}$, and to consider ζ and z sometimes as functions of φ and ξ rather than of φ and ψ. The variable ξ appears continually throughout the entire analysis and seems to be a natural physical variable of the boundary-layer problem. Its close relationship to the variable $\frac{y}{\sqrt{x}}$ which gives the Blasius similarity law is quite evident. It should be noted that inflection points in the ζ, ψ plane correspond identically to inflection points in the ζ, ξ plane, so that our preceding discussion is valid in either plane.

The first procedure investigated was an iteration one, i.e., $\left(\frac{\partial\zeta}{\partial\varphi}\right)^*$ was taken from the first approximation as $\frac{\partial\zeta_{i_1}}{\partial\varphi}$. This had the effect of moving the inflection point in from infinity, but in the cases considered still gave $\zeta_{i_i}>z_0$, i.e., the inflection point still lay outside of the boundary layer. This method was, therefore, abandoned and in all the succeeding investigations the inflection or joining point was forced to occur in the neighborhood of the wall through the choice of $\left(\frac{\partial\zeta}{\partial\varphi}\right)^*$. At the wall we must have $\left(\frac{\partial\zeta}{\partial\varphi}\right)^*=0$ since $\zeta(\psi=0)=0$ for all φ's, while at the joining point we see from equation (24) that $\frac{1}{z_0'}\left(\frac{\partial\zeta}{\partial\varphi}\right)^*_j=1$. Hence a series of simple assumptions for $\left(\frac{\partial\zeta}{\partial\varphi}\right)^*$ which satisfied these two conditions was discussed. The various attempts will be more clear if we restate the immediate problem of the inner solution a little more explicitly. We have a second order differential equation (24) for ζ_i as function of ψ (or ξ), φ being considered constant. We should like to impose the following boundary conditions on the solution of this equation:

At the wall, $\psi=0$, $\zeta_i=0$.

At the joining point, $\psi=\psi_j$ (determined from the outer solution).

$$\zeta_{i_j}=\zeta_{\omega_j} \text{ and } \left(\frac{\partial\zeta_i}{\partial\psi}\right)_j=\left(\frac{\partial\zeta_\omega}{\partial\psi}\right)_j.$$

In addition, we must choose $\left(\frac{\partial\zeta}{\partial\varphi}\right)^*$ so that at the joining point $\left(\frac{\partial^2\zeta_i}{\partial\psi^2}\right)_j=0$. We have here three boundary conditions for a second-order differential equation and must, therefore, abandon one. The condition at the wall is of essential importance, especially for the separation phenomenon, and must be retained. Of the three remaining possibilities two have been discussed. First, the condition $\zeta_{i_j}=\zeta_{\omega_j}$ was abandoned. Second, the joining point was considered to be determined by the condition that $\zeta_{i_j}=\zeta_{\omega_j}$ and the value of ψ_{i_j} was not required to be the same as that for the outer solution, i.e., in general, $\psi_{i_j}\neq\psi_{\omega_j}$. The discarding of the condition on $\left(\frac{\partial\zeta_i}{\partial\psi}\right)_j$ was not considered, largely because the second method was felt to give satisfactory results. When the first method was used $\left(\frac{\partial\zeta}{\partial\varphi}\right)^*$ was assumed to to be given by $\frac{1}{z_0'}\left(\frac{\partial\zeta}{\partial\varphi}\right)^*=\frac{\psi}{\psi_j}$. For the second method two assumptions with regard to $\left(\frac{\partial\zeta}{\partial\varphi}\right)^*$ were considered: $\frac{1}{z_0'}\left(\frac{\partial\zeta}{\partial\varphi}\right)^*=\frac{\zeta_i}{\zeta_j}$ and $\frac{1}{z_0'}\left(\frac{\partial\zeta}{\partial\varphi}\right)^*=\sqrt{\frac{\zeta_i}{\zeta_j}}$. The first is the more natural, but it unfortunately leads to the appearance of elliptic integrals which make the succeeding calculations very awkward. The second is very fortunate in that the inner solution is readily obtained in explicit form and involving only elementary functions.

In order to discuss these various alternatives a family of simple external flows of the form $U^2=b_o+b_1\varphi$ was used. Inaccuracies in the different procedures would be expected to reveal themselves most strikingly at the separation point. Accordingly, the separation point location and the corresponding velocity profile were calculated for all three of the cases mentioned in the preceding paragraph, and all three procedures were found to give very similar results. The calculation of separation point locations followed methods analogous to that described in the next section. In view of this lack of sensitivity of the separation point characteristics to variations in the method of joining, the second assumption of the second method was adopted, i.e.

$$\psi_{i_j}\neq\psi_{\omega_j},\ \frac{1}{z_0'}\left(\frac{\partial\zeta}{\partial\varphi}\right)^*=\sqrt{\frac{\zeta_i}{\zeta_j}}.$$

Before giving the analysis and results for this procedure we shall briefly outline the essential elements of the method finally adopted. For a given value of φ we determine ζ_{ω_j}, $\left(\frac{\partial\zeta_\omega}{\partial\psi}\right)_j$ and ψ_{ω_j} from the outer solu-

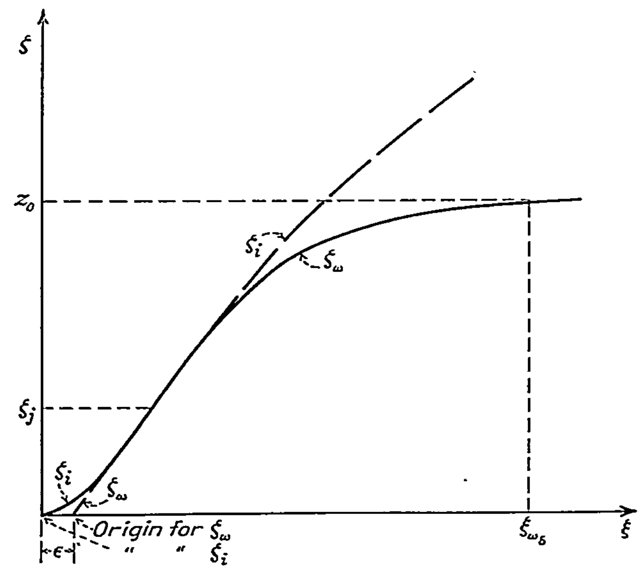

FIGURE 4.—Method of joining inner and outer solutions to obtain the complete joined solution.

tion, using the condition $\left(\frac{\partial^2\zeta_\omega}{\partial\psi^2}\right)_j=0$. We then assume $\frac{1}{z_0'}\left(\frac{\partial\zeta}{\partial\varphi}\right)^*=\sqrt{\frac{\zeta_i}{\zeta_j}}$ and determine the inner solution using the boundary conditions

$$\zeta_i=0 \text{ at } \psi=0,\ \left(\frac{\partial\zeta_i}{\partial\psi}\right)_j=\left(\frac{\partial\zeta_\omega}{\partial\psi}\right)_j \text{ at } \zeta_i=\zeta_{\omega_j}=\zeta_j.$$

We then determine ψ_{i_j} corresponding to ζ_j and define a quantity ϵ by the relation

$$\psi_{i_j}=\psi_{\omega_j}+\sqrt{\varphi}\,\epsilon(\varphi) \text{ or } \xi_{i_j}=\xi_{\omega_j}+\epsilon.$$

For $\zeta<\zeta_j$ we use the inner solution as a function of φ and ξ_i. For $\zeta>\zeta_j$ we use the outer solution, but for a given value of ζ_ω we use, not ξ_ω or ψ_ω but $\xi_\omega+\epsilon$ or $\psi_\omega+\epsilon\sqrt{\varphi}$. Our final approximate solution then has the form indicated schematically in figure 4 by the solid line. This procedure implies that the outer solution is one which does not correspond to the actual wall but to a fictitious one displaced out into the fluid by an amount determined by ϵ. This displacement is a function of the distance downstream from the stagnation point, and in general, has its largest value at the separation point. The error introduced through the use of this displaced outer solution will naturally depend on the relative size of ϵ, increasing as ϵ increases. In the 13 velocity profiles $U^2(\varphi)$ which have been investigated using this procedure the value of ξ_δ (corresponding to the boundary-layer thickness) at the separation point always lay between 1.5 and 2.0 while the largest value of ϵ was 0.026. Hence the displacement was at most of the order of 1.5 percent of the width of the boundary-layer profile. It is, therefore, believed that the error introduced by the displacement of the origin of the outer solution is of negligible importance. In this connection one further procedure which was investigated should be mentioned. In this a slightly altered value of z_0 was taken for the outer solution and the alteration was taken to be such that ϵ vanished. This implies that a solution was obtained for a slightly distorted external potential flow, but the solution was logically consistent. The results with this method were practically indistinguishable from those obtained with the ϵ procedure, so that this more complex method was not carried further.

12. ANALYSIS AND GENERAL RESULTS FOR THE FINAL JOINED SOLUTION

We first derive the expression for the inner solution finally used. Substituting $\frac{1}{z_0'}\left(\frac{\partial\zeta}{\partial\varphi}\right)^*=\sqrt{\frac{\zeta_i}{\zeta_j}}$ in equation (24), we have

$$\frac{\partial^2\zeta_i}{\partial\psi^2}=\frac{A(\varphi)}{\sqrt{\zeta_i}}\left(1-\sqrt{\frac{\zeta_i}{\zeta_j}}\right)$$

Integrating in the same manner as before, we obtain:

$$\frac{\partial\zeta_i}{\partial\psi}=\sqrt{B_1^2+4A\sqrt{\zeta_i}-\frac{2A}{\sqrt{\zeta_j}}\zeta_i}.$$

where, as before,

$$B_1^2=\left(\frac{\partial\zeta_i}{\partial\psi}\right)^2_{\psi=0}$$

Integrating once again and introducing the boundary condition $\zeta_i(\psi=0)=0$, we have

$$\psi=\frac{\sqrt{\zeta_j}}{A}\left[B_1-\sqrt{B_1^2+4A\sqrt{\zeta_i}-\frac{2A}{\sqrt{\zeta_j}}\zeta_i}\right]$$
$$+\sqrt{\frac{2}{A}}\zeta_j^{3/4}\left[\sin^{-1}\frac{1}{\sqrt{1+\frac{B_1^2}{2A\sqrt{\zeta_j}}}}-\sin^{-1}\frac{1-\sqrt{\frac{\zeta_i}{\zeta_j}}}{\sqrt{1+\frac{B_1^2}{2A\sqrt{\zeta_j}}}}\right]$$

It is convenient to rewrite these expressions in terms of the variable ξ. Replacing A by its original expression from (24) a little calculation gives for the final inner solution expressions:

$$\left.\begin{aligned}
\xi_i&=\frac{1}{\sqrt{2}}\sqrt{\frac{z_0}{-z_0'\varphi}}\left(\frac{\zeta_j}{z_0}\right)^{3/4}\left[B-\sqrt{B^2+2\sqrt{\frac{\zeta_i}{\zeta_j}}-\frac{\zeta_i}{\zeta_j}}\right.\\
&\qquad\left.+\sin^{-1}\frac{1}{\sqrt{1+B^2}}-\sin^{-1}\frac{1-\sqrt{\frac{\zeta_i}{\zeta_j}}}{\sqrt{1+B^2}}\right]\\
\frac{\partial\frac{\zeta_i}{z_0}}{\partial\xi}&=2\sqrt{2}\sqrt{\frac{-z_0'\varphi}{z_0}}\left(\frac{\zeta_j}{z_0}\right)^{1/4}\sqrt{B^2+2\sqrt{\frac{\zeta_i}{\zeta_j}}-\frac{\zeta_i}{\zeta_j}}\\
\frac{\partial^2\frac{\zeta_i}{z_0}}{\partial\xi^2}&=\frac{-4z_0'\varphi}{z_0}\left(\frac{\zeta_i}{z_0}\right)^{-1/2}\left(1-\sqrt{\frac{\zeta_i}{\zeta_j}}\right);\ \xi=\frac{\psi}{\sqrt{\varphi}}
\end{aligned}\right\}\tag{29}$$

where

$$B^2=\frac{\left(\frac{\partial\zeta_i}{\partial\psi}\right)^2_{\psi=0}}{-8z_0'z_0\sqrt{\frac{\zeta_j}{z_0}}}\sim\left(\frac{\partial u}{\partial y}\right)^2_{y=0}\tag{30}$$

We obtain the following expressions, giving the inner solution joining-point characteristics, by putting $\zeta_i=\zeta_j$ in (29):

$$\left.\begin{aligned}
\xi_{i_j}&=\frac{1}{\sqrt{2}}\sqrt{\frac{z_0}{-z_0'\varphi}}\left(\frac{\zeta_j}{z_0}\right)^{3/4}\left[B-\sqrt{1+B^2}\right.\\
&\qquad\left.+\sin^{-1}\frac{1}{\sqrt{1+B^2}}\right]\\
\left(\frac{\partial\frac{\zeta_i}{z_0}}{\partial\xi}\right)_j&=2\sqrt{2}\sqrt{\frac{-z_0'\varphi}{z_0}}\left(\frac{\zeta_j}{z_0}\right)^{1/4}\sqrt{1+B^2}
\end{aligned}\right\}\tag{31}$$

To join to the outer solution for any given value of φ, we first determine $\xi_\omega, \frac{\zeta_{\omega j}}{z_0}$, and $\left(\frac{\partial\frac{\zeta_\omega}{z_0}}{\partial\xi}\right)_j$ from the outer solution (17) or (19) using the relation $\left(\frac{\partial^2\frac{\zeta_\omega}{z_0}}{\partial\xi^2}\right)_j=\left(\frac{\partial^2 z_\omega}{\partial\xi^2}\right)_j=\left(\frac{\partial^2 z_\omega}{\partial\psi^2}\right)_j=0$. We then require $\left(\frac{\zeta_i}{z_0}\right)_j=\left(\frac{\zeta_\omega}{z_0}\right)=\frac{\zeta_j}{z_0}, \left(\frac{\partial\frac{\zeta_i}{z_0}}{\partial\xi}\right)_j=\left(\frac{\partial\frac{\zeta_\omega}{z_0}}{\partial\xi}\right)_j$. The second equation of (31) then determines B and this, with the preceding equation, gives ξ_{i_j}. Finally, we obtain ϵ from

$$\epsilon=\xi_{i_j}-\xi_{\omega_j}\tag{32}$$

We now have the complete and joined outer and inner solutions for a given φ. The analysis may then be repeated for as many values of φ as are desired. The fact that the inner solution is given in the form $\xi=\xi(\zeta_i)$, and that inversion to $\zeta_i(\xi)$ is very difficult, is of no practical importance, since what we finally want is the velocity profile $u=u(y)$ for an arbitrary value of x, i.e., of φ. For considering only the inner solution for the moment, we may write the basic relations (9) in the form

$$u=U\sqrt{\frac{\zeta}{z_0}},\ y=\frac{2\sqrt{\varphi}}{U\sqrt{R}}I_\xi,\ I_\xi=\int_0^\xi\frac{d\xi}{\sqrt{\frac{\zeta_i}{z_0}}}$$

For a given φ, ζ_i is given implicitly as a function of ξ in (29). Transforming the integral in I_ξ, so that $\frac{\zeta_i}{\zeta_j}$ is the variable of integration, the quadrature can be analytically carried out and we readily obtain

$$I_\xi=\frac{1}{\sqrt{2}}\sqrt{\frac{z_0}{-z_0'\varphi}}\left(\frac{\zeta_j}{z_0}\right)^{1/4}\left[\sin^{-1}\frac{1}{\sqrt{1+B^2}}-\sin^{-1}\frac{1-\sqrt{\frac{\zeta_i}{\zeta_j}}}{\sqrt{1+B^2}}\right]$$

This equation is easily inverted, so that, using the preceding expressions, we obtain $\sqrt{\zeta_i}=f(y)$ or $u_i=f(y)$ explicitly, as has been done in (33) below.

In collecting the most important expressions for the final, complete solution, it is desirable to include the expression for the so-called "boundary-layer Reynolds Number" R_δ. If U_1 is the actual (dimensional) potential velocity just outside of the boundary layer,

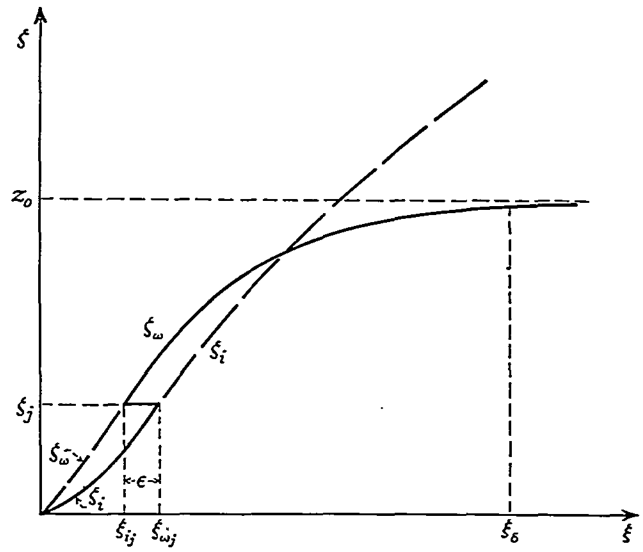

FIGURE 5.—Alternative picture of the method of joining inner and outer solutions.

and δ_1 is the corresponding actual boundary-layer thickness, then R_δ is defined by

$$R_\delta=\frac{U_1\delta_1}{\nu}$$

In presenting the final results it is also convenient to use a slightly different picture from that discussed in connection with figure 4. For conceptual purposes the displaced origin for the outer solution seems most suitable to the authors, since it gives a continuous final solution in the $\zeta(\xi)$ or $\zeta(\psi)$ plane. The notation for the corresponding expressions giving results in the $u(y)$ plane is, however, a little complicated. The alternative picture indicated in figure 5 gives identical results to that of figure 4 but leads to simpler expressions for u and y. In figure 5 we have made the origins for ζ_i and ζ_ω coincide so that we have only one ξ variable instead of the two used formerly (ξ_i and ξ_ω). At the joining point ζ_j, there is now a discontinuity of magnitude ϵ between the values of ξ for the inner and outer solutions ($\xi_{ij}-\xi_{\omega j}=\epsilon$). The complete solution is obtained by following the inner solution curve from the origin to ζ_j and the outer from ζ_j to z_0. (cf. the solid curve.) The solution $u(y)$ is then continuous although $\zeta(\xi)$ is discontinuous, as is clear from the formulas given just below.

The essential results of the entire foregoing analysis may be summarized as follows: For any given value of φ between the stagnation point origin and the separation point:

$$\left.\begin{aligned}
&\text{Writing } y^*=y\sqrt{R}\sqrt{-z_0'},\ \frac{u_j}{U}=\sqrt{\frac{\zeta_j}{z_0}},\text{ then}\\
&\quad\text{for } \zeta\leqq\zeta_j,\text{ i.e., } u\leqq u_j:\\
&\quad\frac{u}{U}=\frac{u_j}{U}\left[1-\sqrt{1+B^2}\sin\left\{\sin^{-1}\frac{1}{\sqrt{1+B^2}}-\frac{y^*}{\sqrt{\frac{u_j}{U}}}\right\}\right]\\
&\quad\text{for } \zeta\geqq\zeta_j,\text{ i.e., } u\geqq u_j:\ \frac{u}{U}=\sqrt{\frac{\zeta_\omega}{z_0}(\xi)}\text{ and } y^*\\
&\quad=\sqrt{\frac{u_j}{U}}\sin^{-1}\frac{1}{\sqrt{1+B^2}}+\sqrt{2}\sqrt{\frac{-z_0'\varphi}{z_0}}\int_{\xi_{\omega j}}^{\xi}\frac{d\xi}{\sqrt{\frac{\zeta_\omega}{z_0}(\xi)}}\\
&\quad y=\delta,\text{ i.e., } y^*=\delta^*\ \textit{for}\ \xi=\xi_\delta\text{ where}\\
&\quad\frac{\zeta_\omega}{z_0}(\xi=\xi_\delta)=0.99\\
&\quad R_\delta=\sqrt{R}\sqrt{\frac{2z_0}{-z_0'}}\delta^*
\end{aligned}\right\}\tag{33}$$

At the separation point, $B=0$, we have the following simplifications:

$$\left.\begin{aligned}
&u\leqq u_j:\ \frac{u}{U}=\frac{u_j}{U}\left(1-\cos\frac{y^*}{\sqrt{\frac{u_j}{U}}}\right)\\
&u\geqq u_j:y^*=\frac{\pi}{2}\sqrt{\frac{u_j}{U}}+\sqrt{2}\sqrt{\frac{-z_0'\varphi}{z_0}}\int_{\xi_{\omega j}}^{\xi}\frac{d\xi}{\sqrt{\frac{\zeta_\omega}{z_0}(\xi)}}
\end{aligned}\right\}\tag{33a}$$

It will be noticed that the boundary-layer thickness $y=\delta$ has been defined in accordance with (21) as that value of y for which $\frac{\zeta_\omega}{z_0}=0.99$, i.e., $\frac{u}{U}=0.995$. This numerical value was chosen because of the excellent agreement it gave with the Blasius expression for δ in the case of flow with constant pressure. It will also be noticed that the final formulation given above completely eliminates the necessity of calculating ξ_{ij} or ϵ.

Before leaving this general analysis of the boundary-layer problem, the method of determining the separation point should be indicated. It would, of course, be possible to determine the value of φ_s giving this point by trial and error, using the procedure already outlined in this section, until a value of φ was found for which $B^2=\left(\frac{\partial u}{\partial y}\right)^2_{y=0}=0$. The problem can, however, be much more easily solved directly by the following method. Since $B^2=0$, we have from (31) for the separation point:

$$\left.\begin{aligned}
&\xi_{ij}=\frac{\frac{\pi}{2}-1}{\sqrt{2}}\sqrt{\frac{z_0}{-z_0'\varphi}}\left(\frac{\zeta_j}{z_0}\right)^{3/4}\\
&\left(\frac{\partial\frac{\zeta_i}{z_0}}{\partial\xi}\right)_j=2\sqrt{2}\sqrt{\frac{-z_0'\varphi}{z_0}}\left(\frac{\zeta_j}{z_0}\right)^{1/4}
\end{aligned}\right\}B^2=0\tag{34}$$

From the outside solution, using the condition that $\left(\frac{\partial^2 z_\omega}{\partial\xi^2}\right)_j=0$, we obtain

$$\left(\frac{\zeta_\omega}{z_0}\right)_j=\mathrm{f}_1(\varphi),\ \left(\frac{\partial\frac{\zeta_\omega}{z_0}}{\partial\xi}\right)_j=\mathrm{f}_2(\varphi),$$

where f_1 and f_2 are known functions of φ. We now employ the conditions:

$$\left(\frac{\zeta_i}{z_0}\right)_j=\left(\frac{\zeta_\omega}{z_0}\right)_j=\frac{\zeta_j}{z_0}\text{ and }\left(\frac{\partial\frac{\zeta_i}{z_0}}{\partial\xi}\right)_j=\left(\frac{\partial\frac{\zeta_\omega}{z_0}}{\partial\xi}\right)_j$$

which, substituted into the second equation of (34), give $\sqrt{\frac{-z_0'\varphi}{z_0}}=\mathrm{f}_3(\varphi)$, where f_3 is also a known function of φ. But z_0 and z_0' are also given functions of φ. We therefore have an equation containing only φ whose solution obviously gives φ_s, i.e., the value of φ at the separation point. In other words, we determine the separation point position φ_s by solving an equation of the form

$$F(\varphi_s)=0.$$

The procedure is entirely similar to that discussed earlier in connection with the use of the unmodified outer solution alone. The calculation of the velocity profile, etc., is carried out in exactly the same manner as was described above in this section for an arbitrary value of φ.

13. APPLICATION TO THE BOUNDARY LAYER FOR A "SINGLE-ROOF PROFILE"

In this section we consider the application of the above analysis to the simplest case of external flow with pressure increase given by $U^2=\beta_0+\beta_1\varphi$. We call this $U^2(\varphi)$ profile a "single-roof profile" for obvious reasons. For any actual flow around a solid body we have an upstream stagnation point, i.e., $U^2=0$ at $\varphi=0$. In order to give our single-roof profile any physical significance we must, therefore, consider it as a limiting case of a "double-roof profile" defined by

$$U^2=b_1\varphi\text{ for }0\leqq\varphi\leqq\varphi_1$$
$$U^2=\beta_0+\beta_1\varphi\text{ for }\varphi\geqq\varphi_1.$$

The limiting single-roof profile is obtained by letting $\varphi_1\to0$, $b_1\to\infty$, while $b_1\varphi_1=U_1^2$. (cf. fig. 6.) The final velocity profile may then be written:

$$U^2 = U_1^2 - \beta\varphi,$$

and may be considered as a first and crude approximation to the flow over the upper surface of an airfoil. To fix ideas we consider this velocity profile as such an approximation. At the trailing edge of an airfoil the velocity has very nearly the value of the undisturbed velocity far from the airfoil. If this be taken as the basic velocity used in defining the Reynolds Number of the problem, then we may take U (trailing edge) $=1$ in accordance with (3). Once again to fix ideas we may consider our airfoil only in the normal range of angles of attack, so that the position of the forward stagnation point varies very little and may be assumed to remain fixed. If we then take the distance along the upper surface from stagnation point

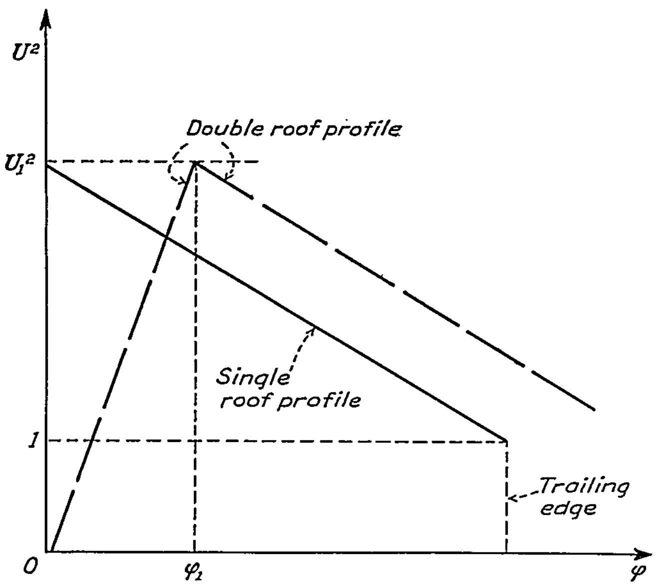

FIGURE 6.—The single-roof potential velocity profile as the limiting case of a double-roof profile.

to trailing edge as the characteristic length of the problem, the dimensionless coordinate x along the surface will have values between zero at the stagnation point and unity at the trailing edge. From the relation between x and φ given in (9), it is then very simple to deduce the function $U(x)$ from $U^2(\varphi)$. The two alternative expressions giving U are finally

$$\left.\begin{aligned} 2z_0 = U^2 &= U_1^2 - 2(U_1-1)\varphi \\ U &= U_1 - (U_1-1)x. \end{aligned}\right\} \tag{35}$$

The corresponding profiles are plotted in figure 7.

In obtaining the outer solution it appears that the results are identical whether we substitute (35) directly into (14) or use the double-roof profile with (17) and then pass to the limit $\varphi_1 \to 0$. In either case, we obtain:

$$\begin{aligned} z_\omega &= U_1^2 g_0(\xi) - 2(U_1-1)[g_0(\xi) + g_1(\xi)]\varphi \\ &= U_1^2 g_0(\xi) - 2(U_1-1)h_1(\xi)\varphi \end{aligned} \tag{36}$$

For the separation criterion, on the other hand, we must use the double-roof profile and the limiting process. If we carry this out using (18), we easily get

$$\varphi_{s_\omega} = \frac{U_1^2}{4(U_1-1)} \tag{37}$$

where the subscript ω is used to indicate the fact that the value of φ_s is obtained from the separation criterion

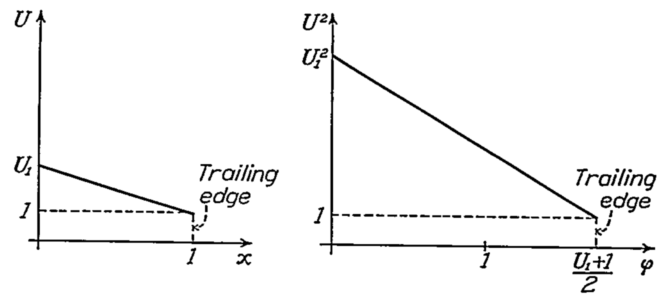

FIGURE 7.—The final single-roof profile as function of x and φ.

for the outer solution only. The functions $g_0(\xi)$, $g_1(\xi)$, and $h_1(\xi)$ are plotted in figure 8.

To carry out the joining procedure with the inner solution, we need $\frac{\zeta_\omega}{z_0} = 1 - \frac{z_\omega}{z_0}$, and in the expression for

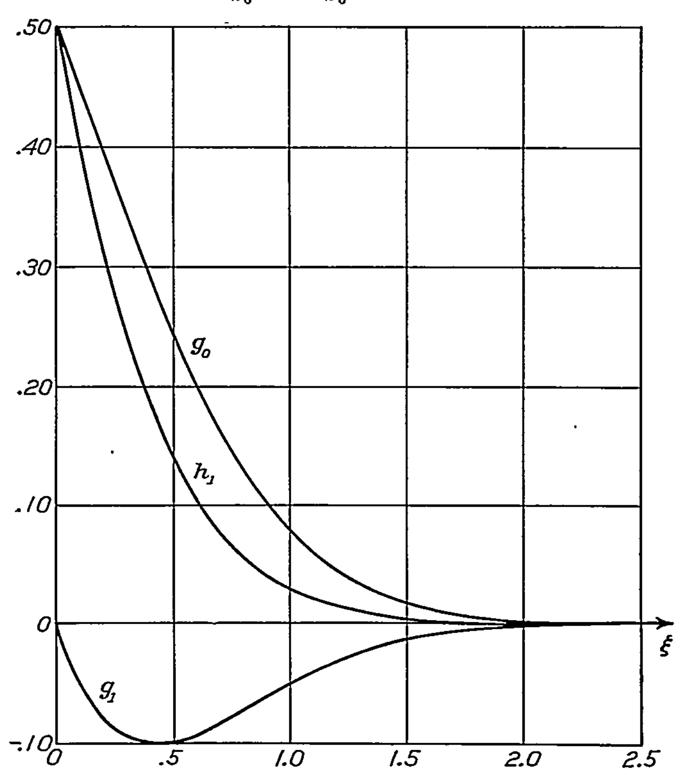

FIGURE 8.—The universal functions g_0; g_1, and h_1.

this quantity it appears convenient to replace φ or U^2 by a new variable denoting position along the surface. We, therefore, write

$$\alpha = \frac{U_1^2}{U^2} = \frac{1}{1 - 2\frac{U_1-1}{U_1^2}\varphi} \tag{38}$$

and obtain at once from (35) and (36)

$$\frac{\zeta_\omega}{z_0}=1-2h_1(\xi)+2\alpha g_1(\xi) \tag{39}$$

From the condition $\frac{\partial^2 \frac{\zeta_\omega}{z_0}}{\partial \xi^2}(\xi=\xi_{\omega_j})=0$, we get $\alpha=\alpha(\xi_{\omega_j})$. Since ξ_{ω_j} is always small, the easiest method which has been discovered for carrying out this calculation in general is to expand the functions $g_i(\xi)$ as power series about $\xi=0$, using the expansion for the probability integral given in section 8. In the present case it was found possible to neglect terms of degree higher than 5 in ξ in the expression for $\frac{\zeta_\omega}{z_0}$, and this is probably also true in general. Having $\alpha=\alpha(\xi_{\omega_j})$, it is easy to calculate $\left(\frac{\zeta_\omega}{z_0}\right)_j$ and $\left(\frac{\partial \frac{\zeta_\omega}{z_0}}{\partial \xi}\right)_j$ from (39) as functions of ξ_{ω_j}.

Turning now to the inner solution, so as to determine the separation point, we obtain from (35) and (38):

$$\frac{-z_0'\varphi}{z_0}=\alpha-1 \tag{40}$$

so that (34) becomes

$$\left(\frac{\partial \frac{\zeta_i}{z_0}}{\partial \xi}\right)_j=2\sqrt{2}\sqrt{\alpha-1}\left(\frac{\zeta_j}{z_0}\right)^{1/4}$$

Now replacing $\left(\frac{\zeta_j}{z_0}\right)^{1/4}$ by $\left(\frac{\zeta_\omega}{z_0}\right)_j^{1/4}$ as obtained in the last paragraph, we have $\left(\frac{\partial \frac{\zeta_i}{z_0}}{\partial \xi}\right)_j=\mathrm{f}(\xi_{\omega_j})$, since $\alpha=\alpha(\xi_{\omega_j})$. Then $\left(\frac{\partial \frac{\zeta_\omega}{z_0}}{\partial \xi}\right)_j-\left(\frac{\partial \frac{\zeta_i}{z_0}}{\partial \xi}\right)_j=F_1(\xi_{\omega_j})$ (say), which may be calculated and plotted. In accordance with the considerations presented at the end of the preceding section, the value of ξ_{ω_j} for which $F_1(\xi_{\omega_j})$ vanishes, is that corresponding to the separation point. The separation point values of α and $\frac{\zeta_\omega}{z_0}$ are given by this value of ξ_{ω_j}. $(\xi_{i_j})_s$ is then determined using (34) and ϵ_s from (32). The numerical values obtained in this way for the "single roof" profile are:

$$\alpha_s=1.241, \left(\frac{\zeta_j}{z_0}\right)_s=0.246,\ (\xi_{\omega_j})_s=0.262,\ \epsilon_s=0.026 \tag{41}$$

To get a picture of the complete boundary-layer behavior, we must consider other values of α between 1 and 1.241, corresponding to points along the surface between the stagnation and separation points. For such intermediate points the procedure is the following: We assume a definite value of α and this, in view of the calculations outlined above, immediately gives $\xi_{\omega_j}, \frac{\zeta_j}{z_0}$, and $\left(\frac{\partial \frac{\zeta_\omega}{z_0}}{\partial \xi}\right)_j$. Substituting the last two into the second expression of (31), remembering that $\left(\frac{\partial \frac{\zeta_\omega}{z_0}}{\partial \xi}\right)_j=\left(\frac{\partial \frac{\zeta_i}{z_0}}{\partial \xi}\right)_j$, we obtain B^2. Although they are not needed for the determination of boundary-layer velocity profiles, we could then calculate ξ_{i_j} and hence ϵ from the first equation of (31).

The final velocity profiles for any of the values of α considered are now determined from (33) or (33a). The integral occurring in these expressions will usually have to be evaluated by an integraph or some other graphical or tabular means. The complete calculations have been carried out for a series of three values of α. In order to visualize the results and to present them in a form independent of the magnitude of U_1, they are given in terms of $\frac{u}{U_1}$, y^*, and x^* where

$$\frac{u}{U_1}=\frac{u}{U}\frac{1}{\sqrt{\alpha}}, y^*=y\sqrt{R}\sqrt{U_1-1}, x^*=x\frac{U_1-1}{U_1}=1-\frac{1}{\sqrt{\alpha}} \tag{42}$$

The velocity profiles and boundary-layer thickness in terms of these variables are indicated in figure 9 by the solid curves. The joining points are shown by the oval symbols with the j. It is very interesting that in the u, y diagram the inner solution is so large a fraction of the complete solution, while in a ζ, ξ plane the outer solution goes nearly to the wall. It is just the small values of ξ at the joining point which make it seem that the approximation of our joined solution to an exact solution of the boundary-layer equations is satisfactory.

For the sake of comparison the analogous calculations have also been carried out using the outer solution only. The results are given by the dashed curves in figure 9. At the separation point as determined from the outer solution (cf. equation (37) the boundary-layer thickness becomes logarithmically infinite since $\frac{\zeta_\omega}{z_0}$ is proportional to ξ^2 near $\xi=0$. This seems to be a general result and indicates the unsuitability of the outer solution alone for the discussion of separation-point characteristics. On the other hand, the outer solution is very close to the joined solution for the first profile (going downstream from the stagnation point), and is not very different for the second profile. It would appear, therefore, that the outer solution alone probably represents a fairly satisfactory approximation in general, as long as the immediate neighborhood of a separation point is not approached.

For the single-roof velocity distribution discussed in this section, the Pohlhausen method of approximate integration of the boundary-layer equations also leads to simple expressions. In deriving these expressions it is again necessary to consider the single-roof profile as a limiting case of a double-roof profile, but this limiting process is very easily carried out, and leads to the following simple form of the Pohlhausen equations:

$$
\left.\begin{aligned}
&\log_e \frac{U_1}{U_1-(U_1-1)x} = \log_e\sqrt{\alpha} = I_\lambda \\
&I_\lambda = \int_\lambda^0 \frac{-213.12+5.76\eta+\eta^2}{-7257.6+1336.32\eta-37.92\eta^2-0.8\eta^3}\,d\eta \\
&\frac{u}{U} = \frac{12+\lambda}{6}\left(\frac{y^*}{\delta^*}\right) - \frac{\lambda}{2}\left(\frac{y^*}{\delta^*}\right)^2 - \frac{4-\lambda}{2}\left(\frac{y^*}{\delta^*}\right)^3 + \frac{6-\lambda}{6}\left(\frac{y^*}{\delta^*}\right)^4 \\
&\delta^* = \sqrt{-\lambda}.
\end{aligned}\right\} \quad (43)
$$

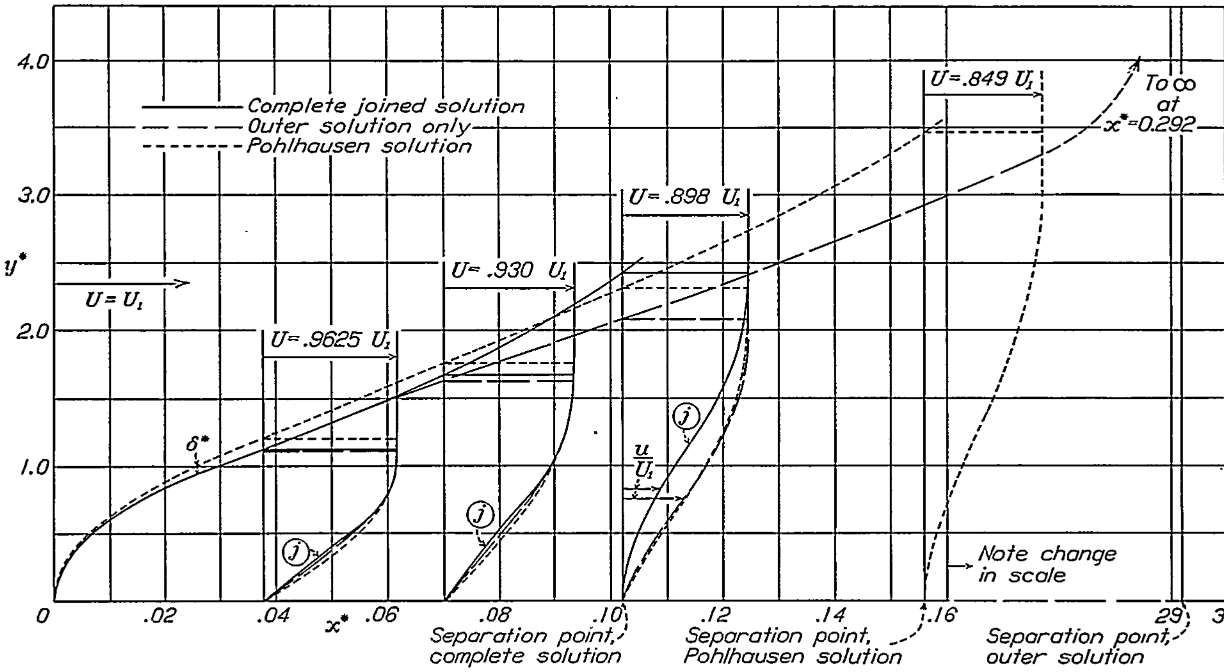

FIGURE 9.—Velocity profiles and boundary-layer thickness for the single-roof potential velocity distribution.

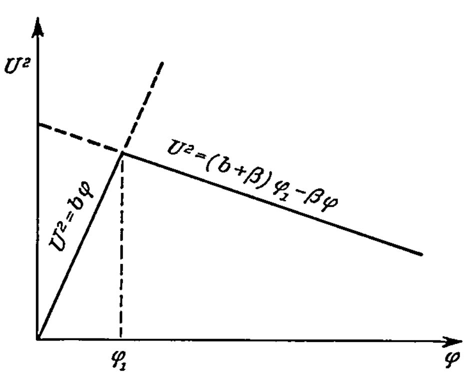

FIGURE 10.—The double-roof profile.

The function $I_\lambda(\lambda)$ may be evaluated analytically or more simply with an integraph. For the separation point $\lambda = -12$ which gives $\alpha_s = 1.404$. Boundary-layer profiles have been calculated from (43) for this value of α as well as for the two values considered above with our joined and outer solutions. The results are plotted in figure 9 as dotted curves. The agreement with our solutions is quite good for the first two profiles, and at the third profile the outer and Pohlhausen solutions are in remarkably good agreement. Since the outer solution is known to be inexact near a separation point, this agreement serves to cast some doubt on the Pohlhausen solution in such a region.

The expressions for R_{δ_s} as calculated by the three methods are:

$$
\left.\begin{aligned}
&R_{\delta_s}\ \text{(joined solution)} = 2.18\,\frac{U_1}{\sqrt{U_1-1}}\sqrt{R} \\
&R_{\delta_s}\ \text{(outer solution)} = \infty \\
&R_{\delta_s}\ \text{(Pohlhausen)} = 2.93\,\frac{U_1}{\sqrt{U_1-1}}\sqrt{R}
\end{aligned}\right\} \quad (44)
$$

14. THE BOUNDARY LAYER FOR A "DOUBLE-ROOF PROFILE"

The results of the last section are particularly simple but the single-roof profile is too restricted to serve as a satisfactory approximation to many potential velocity distributions. The double-roof profile, involving one more free parameter, is considerably more flexible, and may have a much wider field of applicability. In this section, therefore, the separation-point characteristics of a double-roof profile are investigated.

The notation adopted is defined in the following equations, and illustrated graphically in figure 10.

$$
\left.\begin{aligned}
&\text{For } 0 \leq \varphi \leq \varphi_1, \quad U^2 = b\varphi \\
&\text{For } \varphi_1 \leq \varphi, \quad U^2 = (b+\beta)\varphi_1 - \beta\varphi
\end{aligned}\right\} \quad (45)
$$

We are primarily interested in separation phenomena and will therefore restrict our considerations to the region $\varphi > \varphi_1$, in which region only can separation occur. From equations (16) and (19) we obtain for the outer solution corresponding to the flow of figure 10:

$$
z_\omega(\varphi, \psi) = \{(b+\beta)\varphi_1 - \beta\varphi\}h_1^* + b\varphi(h_1 - h_1^*),
$$

where $h_1=h_1\left(\frac{\psi}{\sqrt{\varphi}}\right)$, $h_1^*=h_1\left(\frac{\psi}{\sqrt{\varphi-\varphi_1}}\right)$, and $h_1(x)$ is defined in (14), (19), and plotted in figure 8. It is convenient to replace z_ω by $\frac{\zeta_\omega}{z_0}$ in the following, where

$$\frac{\zeta_\omega}{z_0}=1-\frac{z_\omega}{z_0};\ z_0=\frac{U^2}{2}=\frac{1}{2}\{(b+\beta)\varphi_1-\beta\varphi\}$$

We get at once $\frac{\zeta_\omega}{z_0}=1-2\left\{h_1^*+\frac{b\varphi}{U^2}(h_1-h_1^*)\right\}$ (46)

As above, we write $\frac{\psi}{\sqrt{\varphi}}=\xi$

so that $h_1=h_1(\xi)$, $h_1^*=h_1\left(\xi\sqrt{\frac{\varphi}{\varphi-\varphi_1}}\right)$

Considering $\frac{\zeta_\omega}{z_0}$ as a function of φ, ξ and restricting our attention for the moment to variation with φ, it appears that there are in (46) two variables: $\frac{b\varphi}{U^2}$ and $\sqrt{\frac{\varphi}{\varphi-\varphi_1}}$. However, the relation which must exist between these two variables is very easy to deduce. For

$$\frac{U^2}{b\varphi}=\frac{(b+\beta)\varphi_1-\beta\varphi}{b\varphi}=\frac{\varphi_1}{\varphi}-\frac{\beta}{b}\frac{\varphi-\varphi_1}{\varphi}$$

$$\therefore 1-\frac{U^2}{b\varphi}=\frac{\varphi-\varphi_1}{\varphi}\left(1+\frac{\beta}{b}\right)$$

or

$$\frac{\varphi}{\varphi-\varphi_1}=\frac{1+\frac{\beta}{b}}{1-\frac{U^2}{b\varphi}}$$

Equation (46) may therefore be rewritten:

$$\left.\begin{aligned}&\frac{\zeta_\omega}{z_0}=1-2\{h_1^*+\vartheta(h_1-h_1^*)\}\\&h_1=h_1(\xi),\ h_1^*=h_1\left(\xi\sqrt{\frac{1+r}{1-\frac{1}{\vartheta}}}\right)\\&\text{where } \vartheta=\frac{b\varphi}{U^2},\ r=\frac{\beta}{b},\text{ and } \xi=\frac{\psi}{\sqrt{\varphi}}\end{aligned}\right\}\quad(47)$$

The variable ϑ, defining position along the φ axis, and the parameter r, giving the ratio of the slopes of the two portions of the double-roof profile, recur continually throughout the succeeding analysis.

The first problem is now to investigate the joining of the above outer solution with the inner solution so as to determine the location of the separation point. Equation (34) gives, for the inner solution at the separation joining point

$$\text{where } \frac{\partial^2\frac{\zeta_i}{z_0}}{\partial\xi^2}=0$$

$$\left.\begin{aligned}&\xi_{i_j}=\frac{0.404}{\sqrt{r\vartheta_s}}\left(\frac{\zeta_j}{z_0}\right)^{3/4}\\&\left(\frac{\partial\frac{\zeta_i}{z_0}}{\partial\xi}\right)_j=2.828\sqrt{r\vartheta_s}\left(\frac{\zeta_j}{z_0}\right)^{1/4}\end{aligned}\right\}\quad(48)$$

$$\text{since } \sqrt{\frac{-z_0'\varphi}{z_0}}=\sqrt{\frac{\beta\varphi}{U^2}}=\sqrt{r\vartheta}$$

To join the two solutions, we must now require that $\left(\frac{\partial^2\frac{\zeta_\omega}{z_0}}{\partial\xi^2}\right)_j=0$ and put $\zeta_{\omega_j}=\zeta_{i_j}=\zeta_j$, $\left(\frac{\partial\frac{\zeta_\omega}{z_0}}{\partial\xi}\right)_j=\left(\frac{\partial\frac{\zeta_i}{z_0}}{\partial\xi}\right)_j$

In view of the fact that ξ_j is small, the most satisfactory method which has been found for proceeding with this calculation is to use the power series expanion of $h_1(x)$ about $x=0$. Using the expansion which is given in section 8 for the probability integral occurring in h_1, the espansion is found to be

$h_1(x)=0.5000-1.1284\,x+x^2-0.3761\,x^3+\ .\ .\ .\ .\ .$

The next term in the series is of degree 5 in x and its neglect has been found to cause no perceptible errors in the results, so that for purposes of calculation the series is cut off after the term in x^3. Substituting this expression for $h_1(x)$ in (47) and putting $\left(\frac{\partial^2\frac{\zeta_\omega}{z_0}}{\partial\xi^2}\right)_j=0$, we obtain $\xi_{\omega_j}=f_1(\vartheta, r)$. This in turn leads to expressions of the form

$$\left(\frac{\zeta_\omega}{z_0}\right)_j=f_2(\vartheta, r)\text{ and }\left(\frac{\partial\frac{\zeta_\omega}{z_0}}{\partial\xi}\right)_j=f_3(\vartheta, r)$$

At the separation point $\vartheta=\vartheta_s$. Using these relations and the second equation of (48), the conditions

$$\zeta_{\omega j}=\zeta_{i_j}=\zeta_j\text{ and }\left(\frac{\partial\frac{\zeta\omega}{z_0}}{\partial\xi}\right)_j=\left(\frac{\partial\frac{\zeta_i}{z_0}}{\partial\xi}\right)_j$$

lead to a relation of the form $\vartheta_s=\vartheta_s(r)$.

From equations (47) and (48) we now determine the separation point values of $\frac{\zeta_j}{z_0}$, $\xi_{\omega j}$, and $\epsilon=\xi_{i_j}-\xi_{\omega_j}$ as functions of the parameter r. The numerical results are indicated by the curves of figure 11.

Before proceeding to the determination of the actual position of the separation point, let us consider for a moment the value of $R_{\delta s}=$ the boundary layer Reynolds Number at the separation point. From equations (33) and (33a) this is given by

$$\left.\begin{aligned}&R_{\delta s}=\sqrt{R}\sqrt{\frac{2U_s^2}{\beta}}\delta_s^*\\&\delta_s^*=\frac{\pi}{2}\left(\frac{\zeta_j}{z_0}\right)^{1/4}+\sqrt{2r\vartheta_s}\int_{\xi_{\omega_j}}^{\xi_\delta}\frac{d\xi}{\sqrt{\frac{\zeta_\omega}{z_0}(\xi,\vartheta_s)}}\\&\frac{\zeta_\omega}{z_0}(\xi=\xi_\delta)=0.99.\end{aligned}\right\}\quad(49)$$

In carrying out the numerical calculations the integral in $\delta_s{}^*$ was evaluated with a Coradi integraph for a series of values of ϑ_s, i.e., a series of values of r.

The essential results of the analysis to this point may now be presented in the form $\vartheta_s=\vartheta_s(r)$ and $\delta^*=\delta^*(r)$; the results are therefore expressed in terms of the single parameter r. In proceeding to the final results (separation-point location and value of R_{δ_s}) it is necessary to introduce one more parameter which we take to be φ_1. From the original relation $U^2=(b+\beta)\varphi_1-\beta\varphi$, and from the definition of ϑ it follows immediately that

$$\varphi=\frac{1+r}{1+r\vartheta}\vartheta\varphi_1,\quad \frac{U^2}{\beta}=\frac{1+r}{r(1+r\vartheta)}\varphi_1.$$

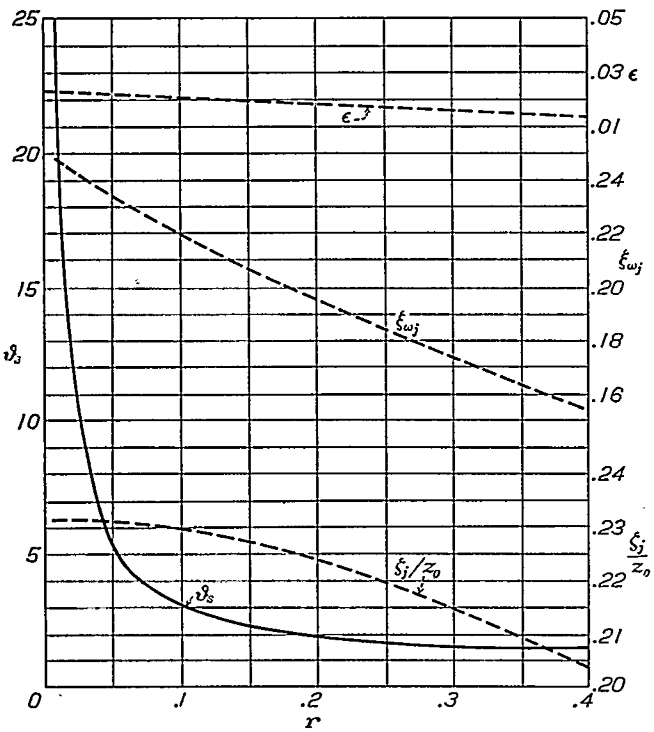

FIGURE 11.—Separation, joining-point characteristics for the double-roof profile.

If we therefore write

$$\left.\begin{aligned}\theta_s&=\frac{1+r}{1+r\vartheta_s}\vartheta_s\\ \Delta_s{}^*&=\sqrt{\frac{2(1+r)}{r(1+r\vartheta_s)}}\delta_s{}^*\end{aligned}\right\}\tag{50}$$

where θ_s and $\Delta_s{}^*$ are functions of r only, then

$$\left.\begin{aligned}\varphi_s&=\varphi_1\theta_s\\ \frac{R_{\delta_s}}{\sqrt{R}}&=\sqrt{\varphi_1}\Delta_s{}^*\end{aligned}\right\}\tag{51}$$

The functions θ_s and $\Delta_s{}^*$ are plotted in figure 12 as functions of $r=\frac{\beta}{b}$. Figure 12 and equations (51) give the position of the separation point (φ_s) and the corresponding value of $\frac{R_{\delta_s}}{\sqrt{R}}$ for any double-roof profile, in terms of the two parameters $r=\frac{\beta}{b}$ and φ_1. The extreme simplicity with which all of the major separation-point characteristics may be determined, for the rather general class of potential flows considered, is quite remarkable.[1]

Two rather interesting results of the above analysis are worthy of special mention. First the small magnitude of the dependence of $\delta_s{}^*$ on r is illustrated by the curve of $\delta_s{}^*$ vs. r which is included in figure 12. In view of the definition of δ^* from equation (33)

$$\delta_s{}^*=\delta_s\sqrt{R}\sqrt{\beta},$$

where δ_s is the actual dimensionless boundary-layer thickness, this nearly constant value of $\delta_s{}^*=2.5$ means that we have

$$\delta_s{}^*\doteq\frac{2.5}{\sqrt{R}}\frac{1}{\sqrt{\beta}}.$$

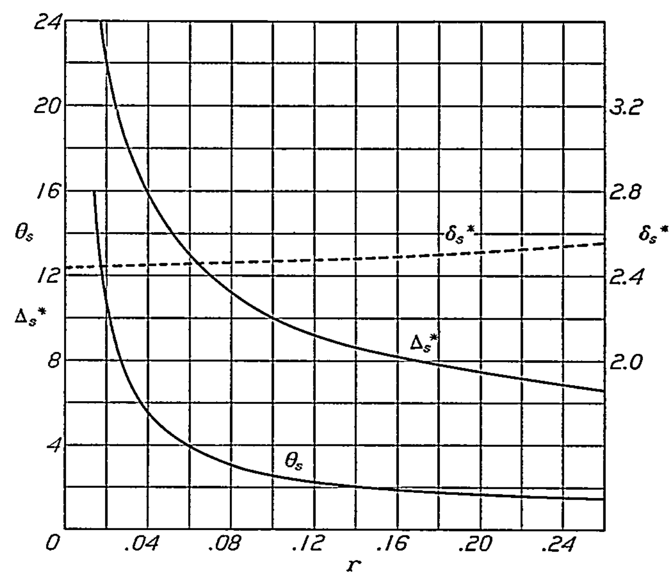

FIGURE 12.—General functions giving separation-point characteristics for the double-roof profile.

In other words, for a given Reynolds Number R, the thickness of the boundary layer at the separation point is approximately inversely proportional to the slope of the second portion of the $U^2(\varphi)$ (or $U(x)$) profile, and independent of the maximum value of U or of the manner in which it is reached, i.e., independent of b or φ_1. The second interesting result lies in the type of simi-

[1] Since θ_s behaves approximately like $\frac{1}{r}$ as r approaches zero, it would appear that the function $r\theta_s$ might have some physical interest. Actually it is easy to show that $\frac{U_s^2}{U_1^2}=1-r(\theta_s-1)$ and this function does have an obvious physical significance. For the limiting case $r=0$, which corresponds to the single-roof profile, we have found $\frac{U_s^2}{U_1^2}=\frac{1}{\alpha_s}=0.806$ (cf. equation 41). As r increases $\frac{U_s^2}{U_1^2}$ increases practically linearly to a value of 0.880 at $r=0.25$. In other words, as the distance along the boundary from the stagnation point to the point of maximum potential velocity increases, the potential velocity at which separation occurs, approaches more and more closely to the maximum velocity. This is exactly the behavior which simple physical considerations would suggest.

larity law which is found to hold for a family of double-roof profiles. For, if we consider a family of such profiles, all of which have the same values of φ_i and of r as in figure 13, then the values of φ_s and of $\frac{R_{\delta_s}}{\sqrt{R}}$ are identical for all the members of the family and for all values of R. This may be considered as an extension of the well-known general boundary-layer theorem which states that for any given potential flow function the position of the separation point along the surface x_s, and the value of $\frac{R_{\delta_s}}{\sqrt{R}}$ are both constant and independent of R.

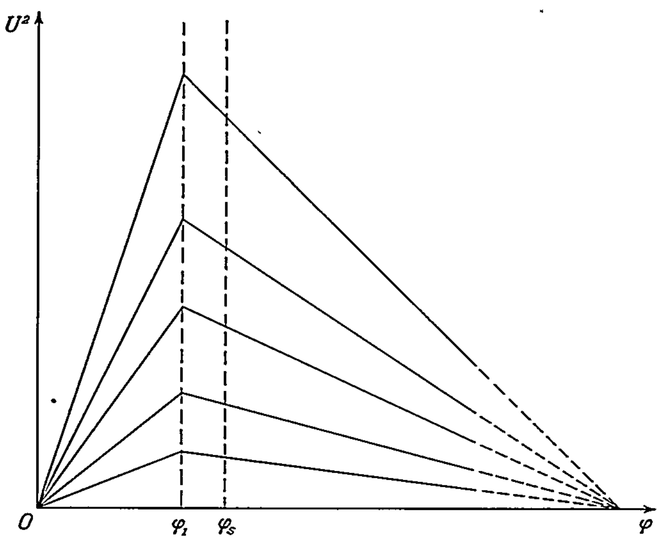

FIGURE 13.—Family of similar potential velocity distributions giving the same values of φ_s and of $R_{\delta_s}/\sqrt{R}$.

CONCLUSION

In the above sections we have developed a new method for determining approximately the characteristics of a laminar boundary layer. General formulas and procedures have been explicitly given enabling the direct application of the method to a rather general class of external flow functions. Two examples are considered in detail and represent approximations to the type of flow which takes place over the upper surface of an airfoil. In most practical cases the boundary layer is turbulent over the main portion of the airfoil section, but the characteristics of the small laminar region downstream from the stagnation point have a great influence on the "transition phenomena" which determine the maximum lift at low and moderate values of Reynolds' Number. In the first example, the calculation of the boundary layer starts from the point where the potential velocity has its maximum value, and the development of the boundary layer and the location of the separation point are investigated in their dependence on the rate at which the velocity decreases with the distance from the point of maximum velocity. In the second example, the calculation starts from the stagnation point and the effects of both the region of increasing velocity and that of decreasing velocity are taken into account. In the latter example, the numerical calculations are restricted to the characteristics of the boundary layer at the separation point.

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